Consider the function

\[ f(x) = \sum_{n=0}^{\infty} \left( \frac{x - 1}{3} \right)^n = 1 + \left( \frac{x - 1}{3} \right) + \left( \frac{x - 1}{3} \right)^2 + \left( \frac{x - 1}{3} \right)^3 + \ldots. \]

What does it mean to define a function by the “infinite polynomial” on the right-hand side of (†)? It means that we can plug in numbers for \( x \) and get numbers back. For instance, \( f(1) = 1 + 0 + 0 + \ldots \), and so \( f(1) = 1 \). What’s \( f(2) \)? We find that

\[
f(2) = 1 + \left( \frac{2 - 1}{3} \right) + \left( \frac{2 - 1}{3} \right)^2 + \left( \frac{2 - 1}{3} \right)^3 + \ldots = 1 + \left( \frac{1}{3} \right) + \left( \frac{1}{3} \right)^2 + \left( \frac{1}{3} \right)^3 + \ldots,
\]

which is a geometric series with initial term \( a \) equal to 1 and the ratio \( r \) equal to \( \frac{1}{3} \); since the absolute value of \( r \) is less than 1, we know that this geometric series converges to \( \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2} \). Thus, \( f(2) = \frac{3}{2} \).

Now, what’s \( f(5) \)? We find that

\[
f(5) = 1 + \left( \frac{5 - 1}{3} \right) + \left( \frac{5 - 1}{3} \right)^2 + \left( \frac{5 - 1}{3} \right)^3 + \ldots = 1 + \left( \frac{4}{3} \right) + \left( \frac{4}{3} \right)^2 + \left( \frac{4}{3} \right)^3 + \ldots,
\]

which is a geometric series with initial term \( a \) equal to 1 and the ratio \( r \) equal to \( \frac{4}{3} \); since \( |r| \geq 1 \), we know that this geometric series diverges, i.e., the sum does not exist. So, \( f(5) \) does not exist – this simply means that 5 is not in the domain of the function \( f \) (recall that the domain of a function is the set of \( x \)’s for which the function is defined).

Thus, defining a function \( f \) as we did in (†) means that the domain of \( f \) is exactly the set of \( x \) values for which the series converges, and the value of \( f \) at any \( x \) in its domain is simply the sum of the series, i.e., the number to which the series converges.

Notice that, by factoring-out the denominators in (†), we can rewrite \( f(x) \) as

\[
(\ast) \quad f(x) = \sum_{n=0}^{\infty} \frac{1}{3^n} (x - 1)^n = 1 + \frac{1}{3}(x - 1) + \frac{1}{3^2}(x - 1)^2 + \frac{1}{3^3}(x - 1)^3 + \ldots.
\]

We think of this as an “infinite polynomial” involving powers of \((x - 1)\), with the coefficients in front of the powers of \((x - 1)\) given by the sequence of reciprocals of powers of 3. To make this precise, for all \( n \geq 0 \), let \( c_n \) = the coefficient in front of \((x - 1)^n\). Then, in the example in (\ast), we have that \( c_0 = 1 \), \( c_1 = \frac{1}{3} \), \( c_2 = \frac{1}{3^2} \), \( c_3 = \frac{1}{3^3} \), and, in general, \( c_n = \frac{1}{3^n} \). Therefore, \( f(x) \) is a function, defined in terms of infinite series, which can be written in the form

\[
(\dagger) \quad f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n = c_0 + c_1 (x - a) + c_2 (x - a)^2 + c_3 (x - a)^3 + \ldots,
\]

where, in our example, \( a = 1 \) and the sequence of coefficients \( c_n \) is determined by the formula \( c_n = \frac{1}{3^n} \).

If we change the value of \( a \) in (\dagger) or if we use another sequence of numbers for the coefficients \( c_n \), we obtain a different function of \( x \) (unless we change them both in a very special way). For instance, if we choose \( a = -5 \) and \( c_n = n^2 \) and put these into the right-hand side of (\dagger), we would obtain a new function, which we will call \( g(x) \), given by

\[
g(x) = \sum_{n=0}^{\infty} c_n (x - a)^n = \sum_{n=0}^{\infty} n^2 (x + 5)^n = 0 + 1(x + 5) + 4(x + 5)^2 + 9(x + 5)^3 + 16(x + 5)^4 + \ldots
\]
Any function given by an expression of the form
\[ \sum_{n=0}^{\infty} c_n(x - a)^n = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \ldots \]
is called a **power series centered at** \( a \), \( a \) is called the **center of the power series**, and the numbers given in the sequence \( c_n \) are called the **coefficients of the power series**. Instead of saying “a power series centered at \( a \)”, we sometimes say “a power series **about** \( a \)”, or “a power series **around** \( a \)”. Thus, the function \( f(x) \) above is a power series centered at 1, while the function \( g(x) \) above is a power series centered at \(-5\).

**What’s so nice about functions that are power series?**

Great. So we can define functions by power series. Why on Earth would we want to?

It turns out that most functions that we deal with are equal to power series, at least, for some values of \( x \). For example,
\[ \ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x - 1)^n = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4 + \ldots, \]
provided that \( 0 < x \leq 2 \); this means that the power series on the right-hand side converges for all numbers \( x \) such that \( 0 < x \leq 2 \) and that the number that the series converges to is precisely equal to the value of \( \ln(x) \).

Big deal. How does that explain why we care about functions given by power series? Why is it nice to think of \( \ln(x) \) as the power series \((x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4 + \ldots\)? Representing a known function, such as \( \ln(x) \), by a power series is nice for two reasons.

The first reason is that, for values of \( x \) that are close to the center, \( a \), the quantity \((x - a)\) will be close to 0, which means that when we raise \((x - a)\) to higher and higher powers, we get smaller and smaller numbers (smaller in absolute value). In other words, if \( x \) is close to \( a \), then \((x - a)^n\) becomes more and more insignificant as \( n \) gets bigger and bigger. This means that, if \( x \) is close to \( a \), the powers of \((x - a)\) that are farther out in the power series – the terms of higher degree – contribute less and less to the sum of the series, assuming the the coefficients in front of these powers do not get very big. Therefore, if \( x \) is close to the center \( a \) of the power series, the entire sum of the infinite series can be approximated well by the first few terms of the series.

For example, since \( \ln(x) = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4 + \ldots \), we should be able to take values of \( x \) near the center 1, and approximate \( \ln(x) \) well by using the polynomial \( p(x) = (x - 1) - \frac{1}{2}(x - 1)^2 \). To eight decimal places, we find that \( \ln(1.1) = 0.09531018 \) while \( p(1.1) = 0.095 \). A pretty good approximation. Moreover, if we take an \( x \) value closer to the center, we expect the approximation to be better. We find \( \ln(1.01) = 0.00995033 \), while \( p(1.01) = 0.00995 - \) accurate to 5 decimal places.

What if we use more terms from our power series for the approximation? Let \( q(x) = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 \). Then, \( q(1.1) = 0.09533333 \) and \( q(1.01) = 0.00995033 \). Looking at the values of \( \ln(1.1) \) and \( \ln(1.01) \) from the previous paragraph, we see that these approximations using \( q(x) \) are accurate to 4 and 8 decimal places, respectively.

This ease of approximating complicated functions by polynomials, and our ability to improve the approximation by using a larger partial sum, is the first reason why we want to find power series representations of functions.

The second reason why we wish to represent function by power series is that, for many purposes, we really can treat power series as though they are simply big polynomials. Therefore, we can differentiate and integrate power
series simply by using the power rules for differentiation and integration over and over on each term of the series. For instance, take our \( g(x) \) from above,

\[
g(x) = 0 + 1(x + 5) + 4(x + 5)^2 + 9(x + 5)^3 + 16(x + 5)^4 + 25(x + 5)^5 + \ldots.
\]

Then, we find that the derivative is

\[
g'(x) = 1+4\cdot 2(x+5)+9\cdot 3(x+5)^2+16\cdot 4(x+5)^3+25\cdot 5(x+5)^4+\ldots = 1+8(x+5)+27(x+5)^2+64(x+5)^3+125(x+5)^4+\ldots,
\]

while the integral is

\[
\int g(x) \, dx = C + \frac{1}{2}(x + 5)^2 + \frac{4}{3}(x + 5)^3 + \frac{9}{4}(x + 5)^4 + \frac{16}{5}(x + 5)^5 + \frac{25}{6}(x + 5)^6 + \ldots.
\]

(However, there is the question: for what \( x \) values are the above equalities true?)

Algebraic manipulations with power series are also easy. Returning to

\[
\ln(x) = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4 + \ldots,
\]

and dividing both sides by \((x - 1)\), we find that

\[
\frac{\ln(x)}{x-1} = 1 - \frac{1}{2}(x - 1) + \frac{1}{3}(x - 1)^2 - \frac{1}{4}(x - 1)^3 + \ldots,
\]

if \(0 < x \leq 2\) and \(x \neq 1\).

We can also substitute into power series to obtain other power series. Later, we shall see that, for all \(x\),

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots.
\]

It follows that we may replace \(x\) by \(-x^2\) on both sides to obtain that, for all \(x\),

\[
e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \ldots.
\]

While you should have heard in Calculus 2 that there is no nice, finite formula for the anti-derivative of \(e^{-x^2}\), it is easy to find a formula using power series; we simply integrate the equation above to obtain

\[
\int e^{-x^2} \, dx = C + x - \frac{x^3}{3} + \frac{x^5}{2! \cdot 5} - \frac{x^7}{3! \cdot 7} + \frac{x^9}{4! \cdot 9} - \ldots.
\]

These are the reasons why we like to have power series representations of functions. Power series can be approximated well, near the center, by the polynomials obtained by taking the partial sums of just a few terms, and power series are easy to manipulate in many ways.

**What are the steps involved in looking at functions in terms of power series?**

1) Given a power series, i.e., given a center \(a\) and a sequence of coefficients \(c_n\), we want to determine its domain – that is, we want to determine where it converges. The domain of a power series is called its **interval of convergence**.

2) We want to manipulate power series: take derivatives, integrals, make substitutions, and perform algebraic operations on power series. We also need to find the new interval of convergence of the manipulated series.
3) Given a known function, \( f(x) \) (such as \( \ln(x) \)) and a center \( a \), we want to find a power series centered at \( a \), \( \sum_{n=0}^{\infty} c_n(x-a)^n \), that equals \( f(x) \). This means that we have to have a way to calculate the coefficients \( c_n \). The power series that we obtain is called the **Taylor series of \( f(x) \) centered at \( a \)**.

4) Once we find the Taylor series of a function \( f(x) \), we want to estimate how well the partial sums of the Taylor series approximate \( f(x) \). These partial sums are called **Taylor polynomials** and the amount of error in approximating \( f(x) \) by its Taylor polynomials is measured by the **error term**.

Sounds like fun, doesn’t it?

**The interval of convergence**

For any power series \( f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \), the domain of the power series – which consists of those \( x \) values for which the series converges – forms an interval, which has \( a \) at its center; this interval is called the **interval of convergence**. The distance from the center \( a \) to either end of the interval of convergence is called the **radius of convergence**, which we will denote by \( R \).

If \( R = \infty \), then the interval of convergence is the entire real line, i.e., the interval \((-\infty, \infty)\); in this case, the series converges for all \( x \). In fact, the series will converge absolutely for all \( x \).

If \( R = 0 \), then the interval of convergence is the closed interval \([a, a]\); in this case, the only place that the series converges is at the center.

If \( 0 < R < \infty \), then the interval of convergence can be of 4 different kinds. It could be the open interval \((a-R, a+R)\), that is, all those \( x \)'s such that \( a-R < x < a+R \), i.e., \( |x-a| < R \). It could be the closed interval \([a-R, a+R]\), i.e., all those \( x \)'s such that \( a-R \leq x \leq a+R \). Or, the interval of convergence could be one of the half-open/half-closed intervals \((a-R, a+R]\) or \([a-R, a+R)\). Regardless of which of these 4 cases we are in, it is always true that the series converges absolutely if \( |x-a| < R \), and diverges if \( |x-a| > R \); the only question is what happens when \( |x-a| = R \), i.e., when \( x = a \pm R \), which occurs at the endpoints of the interval.

To give an example, suppose that \( a = 1 \) and \( R = 3 \). Then, the interval of convergence would be one of \((-2, 4)\), \([-2, 4]\), \((-2, 4)\), and \([-2, 4]\).

To find the radius of convergence, \( R \), one (almost) always uses the ratio test. If \( R \) turns out to be \( \infty \) or 0, then that immediately tells us the interval of convergence. If \( 0 < R < \infty \), then – in order to completely determine the interval of convergence – one must examine whether the endpoints \( a-R \) and \( a+R \) are points where the series converges.

Let’s do an example. Take the series

\[
h(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n} (x-1)^n = \frac{(x-1)}{2} - \frac{1}{2 \cdot 2^2} (x-1)^2 + \frac{1}{3 \cdot 2^3} (x-1)^3 - \frac{1}{4 \cdot 2^4} (x-1)^4 + \ldots
\]

We want to perform the ratio test with the variable \( x \) still in the terms. Let

\[
L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n+1-1}}{(n+1)2^{n+1}} (x-1)^{n+1} \right| = \lim_{n \to \infty} \left| \frac{n}{n+1} \cdot \frac{2^n}{2^{n+1}} (x-1) \right| = \left| \frac{x-1}{2} \right|
\]

Then, the ratio test says that the series converges absolutely if \( L < 1 \), diverges if \( L > 1 \), and tells us nothing when
L = 1. Thus, in this example, the ratio test tells us that the series converges absolutely if \( \frac{|x-1|}{2} < 1 \) and diverges if \( \frac{|x-1|}{2} > 1 \). So, what’s the radius of convergence?

As we discussed above, the radius of convergence is that number \( R \) such that the series converges absolutely if \(|x-a| < R\), and diverges if \(|x-a| > R\). Therefore, to determine \( R \), we have to write the inequalities from the ratio test in the form \(|x-a|\) is less than or greater than something, and then that something is the radius of convergence. In our example, the series converges absolutely if \( \frac{|x-1|}{2} < 1 \), i.e., if \(|x-1| < 2\), and diverges if \( \frac{|x-1|}{2} > 1 \), i.e., if \(|x-1| > 2\). Thus, the radius of convergence is \( R = 2 \), and the interval of convergence is one of \((-1, 3)\), \([-1, 3]\), \((-1, 3]\), or \([-1, 3)\).

Note that the endpoints of the interval of convergence are exactly where the ratio test would give \( L = 1 \). Because of this the ratio test (and the root test) never tell us whether the endpoints are in the interval of convergence; you would need to use some other test to decide. What do you do? You stick \(-1\) and \(3\) into the original series, and use some other test.

So, what’s \( h(-1) \)? Plugging in, we find that

\[
h(-1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n} (-2)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n} (-1)^n \cdot 2^n = \sum_{n=1}^{\infty} \frac{-1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n},
\]

which is negative the harmonic series, and so, diverges. Therefore, \(-1\) is not in the interval of convergence.

What about the other endpoint, \( x = 3 \)? We find that

\[
h(3) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n} 2^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n},
\]

which is the alternating harmonic series, and so, converges conditionally. Thus, \( 3 \) is in the interval of convergence, and now we finally know that the interval of convergence is precisely \((-1, 3]\).

Note that, if a series converges at one or both of the endpoints of the interval of convergence, it may do so conditionally or absolutely, but remember that, for all \( x \) in the open interval \((a - R, a + R)\), there is no question – the series converges absolutely.