THE NEXUS DIAGRAM AND INTEGRAL RESTRICTIONS ON THE MONODROMY

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Abstract. Given a complex analytic function with a one-dimensional critical locus at the origin, we use a new device – the nexus diagram – to examine the monodromy action on the integral cohomology of the Milnor fiber. The nexus diagram relates this monodromy to that of a generic hyperplane slice through the origin, and to that of a generic hyperplane slice near the origin. We thereby obtain number-theoretic restrictions on the monodromy and on the cohomology of the original Milnor fiber.

§1. **Introduction**

Let $\mathcal{U}$ be an open neighborhood of the origin in $\mathbb{C}^{n+1}$, and let z_0 be a non-zero linear form on $\mathbb{C}^{n+1}$. Let $f : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ be a complex analytic function with a 1-dimensional critical locus at the origin, i.e., $\dim_{\mathbf{0}}\Sigma f = 1$. We assume that $\mathcal{U}$ is chosen small enough so that $\Sigma f \subseteq V(f)$.

Let $H_0 := \mathcal{U} \cap V(z_0)$ and let $H_t := \mathcal{U} \cap V(z_0 - t)$, where $t \neq 0$ is sufficiently small (relative to $\mathcal{U}$). Let $f_0 := f_{|_{H_0}}$ and $f_t := f_{|_{H_t}}$. We assume that z_0 is generic enough, and the $\mathcal{U}$ is small enough, so that $\Sigma(f_0) = \{\mathbf{0}\}$. Assuming again the $\mathcal{U}$ is small enough, this implies that $\Sigma(f_t)$ consists of a finite number of points: the points of $H_t \cap \Sigma f$, and the points where H_t intersects the relative polar curve Γ^1_{f,z_0}.

Let F denote the Milnor fiber of f at $\mathbf{0}$, and let $m_{n-1} : \widetilde{H}^{n-1}(F;\ \mathbb{Z}) \xrightarrow{\cong} \widetilde{H}^{n-1}(F;\ \mathbb{Z})$ and $m_n : \widetilde{H}^{n}(F;\ \mathbb{Z}) \xrightarrow{\cong} \widetilde{H}^{n}(F;\ \mathbb{Z})$ denote the corresponding f-monodromy actions on reduced integral cohomology. Note that $\widetilde{H}^{n-1}(F;\ \mathbb{Z})$ is free-Abelian. Let F_0 denote the Milnor fiber of f_0 at $\mathbf{0}$, and let

$$h_0 : \mathbb{Z}^{\mu_0} \cong \widetilde{H}^{n-1}(F_0;\ \mathbb{Z}) \xrightarrow{\cong} \widetilde{H}^{n-1}(F_0;\ \mathbb{Z}) \cong \mathbb{Z}^{\mu_0}$$

denote the corresponding f_0-monodromy action, where μ_0 denotes the Milnor number of the isolated critical point. For each point $\mathbf{p}_i \in H_t \cap \Sigma f$, there is an associated Milnor fiber F_i of f_t at $\mathbf{p}_i$, together with the associated f_t-monodromy action on

$$h_i : \mathbb{Z}^{\mu_i} \cong \widetilde{H}^{n-1}(F_i;\ \mathbb{Z}) \xrightarrow{\cong} \widetilde{H}^{n-1}(F_i;\ \mathbb{Z}) \cong \mathbb{Z}^{\mu_i}.$$

The monodromies of f restricted to H_0 and H_t are compatible with the monodromy of f itself; they are all determined by letting the value of f move in a small circle around the origin in $\mathbb{C}$. In categorical terms, this compatibility is a result of the fact that the monodromy is a natural automorphism of the vanishing cycle functor ϕ_f.

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This compatibility enables us to produce a diagram on which the monodromy acts commutatively

$$\begin{array}{ccccc} & & \bigoplus_i \mathbb{Z}^{\mu_i} & & \\ & & \Big\downarrow{\scriptstyle \alpha} & & \\ \mathbb{Z}^{\mu_0} & \xrightarrow{\ \gamma\ } & N & \xrightarrow{\ \delta\ } & \mathbb{Z}^{\lambda^0_f} \\ & & \Big\downarrow{\scriptstyle \beta} & & \\ & & \mathbb{Z}^{\omega}, & & \end{array}$$

where both the row and the column are short exact (we have omitted the zeroes on each end),

$$\lambda^0_f := \left(\Gamma^1_{f,z_0} \cdot V\left(\frac{\partial f}{\partial z_0}\right)\right)_{\mathbf{0}}$$

is the 0-th Lê number of f (with respect to z_0 at the origin), and $\omega := \big(\Gamma^1_{f,z_0} \cdot V(f)\big)_{\mathbf{0}}$. It is important, and well-known, that $\omega \geqslant \lambda^0_f$, with equality if and only if $\omega = \lambda^0_f = 0$. In addition, by definition, the 1-dimensional Lê number $\lambda^1_f := \sum_i \mu_i$, so that the top node of the diagram is isomorphic to $\mathbb{Z}^{\lambda^1_f}$.

It is also important to note that, by the Lefschetz number result of A'Campo [**1**], the trace of of the monodromy maps h_0 and h_i are all equal to $(-1)^n$.

We refer to the N in the middle of the diagram as the *nexus*, and refer to the whole diagram as the *nexus diagram*. In fact, it is fairly unimportant what the nexus of the diagram is; what is important is that there exists such a diagram on which the f-monodromy acts.

Let us denote the pull-back via α and γ by P, and the push-forward via β and δ by Q. Then, it is trivial to show that

$$P \ \cong\ \ker(\beta \circ \gamma) \ \cong\ \ker(\delta \circ \alpha)$$

and

$$Q \ \cong\ \operatorname{coker}(\beta \circ \gamma) \ \cong\ \operatorname{coker}(\delta \circ \alpha)$$

This is important because $\beta\circ\gamma$ is the map induced on cohomology by the Morse-theoretic attaching map given by Lê in [**3**]. This implies that $\ker(\beta \circ \gamma) \cong \widetilde{H}^{n-1}(F;\ \mathbb{Z})$ and $\operatorname{coker}(\beta \circ \gamma) \cong \widetilde{H}^{n}(F;\ \mathbb{Z})$. An alternate way of seeing these isomorphisms is in terms of Lê numbers (see [**7**]); the map $\delta\circ\alpha : \bigoplus_i \mathbb{Z}^{\mu_i} \cong \mathbb{Z}^{\lambda^1_f} \to \mathbb{Z}^{\lambda^0_f}$ is precisely the integral version of the one non-trivial map appearing in the chain complex of Corollary 10.10 of [**7**] (though there is a typographically error in this corollary – the arrows are reversed). As the cohomology of this complex yields the reduced cohomology of F, we again conclude that $P \cong \widetilde{H}^{n-1}(F;\ \mathbb{Z})$ and $Q \cong \widetilde{H}^{n}(F;\ \mathbb{Z})$.

However, it is not only the f-monodromy that acts on the nexus diagram. We shall see that the z_0-monodromy also acts commutatively on the diagram, acting as the identity on the $\mathbb{Z}^{\mu_0}$ and $\mathbb{Z}^{\omega}$ nodes. In addition, since the z_0-monodromy is also a natural isomorphism, it follows that the actions of the f-monodromy and z_0-monodromy on the nexus diagram commute with each other.

To continue our discussion, we must adopt some notation for the z_0-monodromy action on $\bigoplus_i \mathbb{Z}^{\mu_i}$. Let ν be a (reduced) component of Σf. Assume that $\mathcal{U}$ is small enough so that ν is homeomorphic to a disk, and small enough so that the vanishing cycles along f, restricted to the punctured disk $\nu^* := \nu - \{\mathbf{0}\}$, form a local system. Then, there is an *internal* (or *vertical*) monodromy action ι_ν induced on a given stalk of the vanishing cycle local system; one moves once around the "hole" in the punctured disk ν^*. (For more on vertical monodromy, see [**12**], [**13**], [**14**], [**15**].)

Now, for each component ν of Σf, there are $k_\nu := (\nu \cdot V(z_0))_{\mathbf{0}}$ points $\mathbf{p}_i$ which occur in $\nu \cap H_t$. At each of these $\mathbf{p}_i$, f_t has the same Milnor number; let us denote this common value by μ_ν. There is a "fractional internal monodromy" action, ${}^i\tau_\nu : \mathbb{Z}^{\mu_\nu} \xrightarrow{\cong} \mathbb{Z}^{\mu_\nu}$ given by moving cyclicly from one $\mathbf{p}_i$ in $\nu \cap H_t$ to the next. Thus, taking $\mathbf{p}_1$ as our base point, $\iota_v = {}^{k_\nu}\tau_\nu \circ \cdots \circ {}^1\tau_\nu$. For each i with $2 \leqslant i \leqslant k_\nu$, we identify the copy of $\mathbb{Z}^{\mu_\nu}$ with the copy at $\mathbf{p}_1$ via the isomorphism ${}^{(i-1)}\tau_\nu \circ \cdots \circ {}^1\tau_\nu$. Then, the z_0-monodromy action on $\displaystyle\bigoplus_{\mathbf{p}_i \in \nu \cap H_t} \mathbb{Z}^{\mu_i}$ is defined by the map $\lambda_\nu : (\mathbb{Z}^{\mu_\nu})^{k_\nu} \to (\mathbb{Z}^{\mu_\nu})^{k_\nu}$ given by

$$\lambda_\nu(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k_\nu}) = (h(\mathbf{v}_{k_\nu}), \mathbf{v}_1, \dots, \mathbf{v}_{k_\nu - 1}),$$

from which one concludes immediately that $\ker(\mathrm{id} - \iota_\nu) \cong \ker(\mathrm{id} - \lambda_\nu)$.

Returning to the nexus diagram, since the z_0-monodromy acts as the identity on $\mathbb{Z}^{\mu_0}$, it follows that $\widetilde{H}^{n-1}(F;\ \mathbb{Z})$ must be contained in $\bigoplus_\nu \ker(\mathrm{id} - \lambda_\nu) \cong \bigoplus_\nu \ker(\mathrm{id} - \iota_\nu)$. In particular, the rank of $\widetilde{H}^{n-1}(F;\ \mathbb{Z})$ is at most $\sum_\nu \mu_\nu$.

Let us use $\mathrm{char}_{\_}(t)$ to denote characteristic polynomials. Then, using our discussion above and the proof in Section 2, what we show is:

Main Theorem. *The nexus diagram exists, and the f-monodromy and z_0-monodromy act commutatively on it, and these actions commute with each other.*

The z_0-mondromy acts as the identity on $\widetilde{H}^{n-1}(F_0;\ \mathbb{Z}) \cong \mathbb{Z}^{\mu_0}$ and on $\widetilde{H}^{n}(F, F_0;\ \mathbb{Z}) \cong \mathbb{Z}^{\omega}$. The kernel of the identity minus the z_0-monodromy on $\bigoplus_i \widetilde{H}^{n-1}(F_i;\ \mathbb{Z}) \cong \bigoplus_i \mathbb{Z}^{\mu_i}$ is isomorphic to $\bigoplus_\nu \ker(\mathrm{id} - \iota_\nu)$.

Therefore, $\widetilde{H}^{n-1}(F;\ \mathbb{Z})$ injects into $\bigoplus_\nu \ker(\mathrm{id} - \iota_\nu)$. Moreover, if h_ν denotes one of the h_i for $\mathbf{p}_i \in \nu$, then $\mathrm{char}_{m_{n-1}}(t)$ divides $\mathrm{char}_{h_0}(t)$ and $\prod_\nu \mathrm{char}_{h_\nu}(t)$ in $\mathbb{Z}[t]$, i.e.,

$$\mathrm{char}_{m_{n-1}}(t) \ \Big|\ \gcd\Big(\mathrm{char}_{h_0}(t),\ \prod_\nu \mathrm{char}_{h_\nu}(t)\Big).$$

The above theorem is closely related to the work of many others, and particularly to the work of Siersma. That the z_0-mondromy acts as the identity on $\widetilde{H}^{n-1}(F_0;\ \mathbb{Z})$ and on $\widetilde{H}^{n}(F, F_0;\ \mathbb{Z})$ was first observed by Siersma in [**12**] and [**13**]. That the kernel of the identity minus the z_0-monodromy on $\bigoplus_i \widetilde{H}^{n-1}(F_i;\ \mathbb{Z}) \cong \bigoplus_i \mathbb{Z}^{\mu_i}$ is isomorphic to $\bigoplus_\nu \ker(\mathrm{id} - \iota_\nu)$ was also discussed in [**12**, p. 193–194]. That $\widetilde{H}^{n-1}(F;\ \mathbb{Z})$ injects into $\bigoplus_\nu \ker(\mathrm{id} - \iota_\nu)$ is closely related to Siersma's variation ladder and Proposition 5.3 of [**13**]. The result that $\mathrm{char}_{m_{n-1}}(t)$ divides $\prod_\nu \mathrm{char}_{h_\nu}(t)$ is a cohomological version of Proposition 5.4 of [**13**]. That $\mathrm{char}_{m_{n-1}}(t)$ divides $\mathrm{char}_{h_0}(t)$ follows from the naturality of the monodromy in the proof of Lê's attaching result in [**3**].

A major point of this paper is that the machinery of the derived category makes the proof of the Main Theorem a short exercise. While we consider this an advantage of our approach, it is true that the formalism of the derived category hides much of the beautiful geometry underlying the results. One should compare this with Siersma's work in [**12**], [**13**], and [**14**], which uses integral homology and topological/geometric arguments; note that Siersma does, in fact, discuss the cohomological/vanishing cycle approach in [**13**, p. 467–468], and [**14**, p. 458].

Aside from proving the Main Theorem, we also give a few applications of it. We show how the nexus diagram simplifies the proof of the non-splitting result of Lê in [**4**]. We show how the Main Theorem allows us to generalize Lê's non-splitting result and obtain the main result of [**6**]. Finally, we show how the Main Theorem generalizes the main result of [**8**] from the case of arrangements of planes in $\mathbb{C}^3$ to the case of arbitrary affine hypersurfaces with 1-dimensional critical loci.

In the final section of the paper, we make some brief remarks which may lead to future applications.

§2. **Proof of the Main Theorem**.

We continue with the notation from the introduction.

Let $k : \mathcal{U} \cap V(z_0) \hookrightarrow \mathcal{U}$, $\hat{k} : V(f) \cap V(z_0) \hookrightarrow V(f)$, and $l : \mathcal{U} - \mathcal{U} \cap V(z_0) \hookrightarrow \mathcal{U}$ denote the inclusions. Let $\hat{z}_0 := z_{0|_{V(f)}}$.

Let $\mathbf{P}^\bullet$ denote the complex of sheaves of $\mathbb{Z}$-modules $\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$; this sheaf is perverse. There is a fundamental distinguished triangle

$$l_!l^!\mathbf{P}^\bullet \rightarrow \mathbf{P}^\bullet \rightarrow k_*k^*\mathbf{P}^\bullet \xrightarrow{[1]},$$

which we can "turn" to yield

$$k_*k^*\mathbf{P}^\bullet[-1] \rightarrow l_!l^!\mathbf{P}^\bullet \rightarrow \mathbf{P}^\bullet \xrightarrow{[1]} .$$

Applying the composed functor $\phi_{\hat{z}_0}[-1]\phi_f[-1]$ to the above triangle, we obtain the distinguished triangle

$$(\dagger) \qquad \phi_{\hat{z}_0}[-1]\phi_f[-1]k_*k^*\mathbf{P}^\bullet[-1] \rightarrow \phi_{\hat{z}_0}[-1]\phi_f[-1]l_!l^!\mathbf{P}^\bullet \rightarrow \phi_{\hat{z}_0}[-1]\phi_f[-1]\mathbf{P}^\bullet \xrightarrow{[1]} .$$

Now,

1) $k^*\mathbf{P}^\bullet[-1] \cong \mathbb{Z}^\bullet_{\mathcal{U}\cap V(z_0)}[n]$, and so $\phi_f[-1]k_*k^*\mathbf{P}^\bullet[-1] \cong \hat{k}_*\phi_{f_0}[-1]\mathbb{Z}^\bullet_{\mathcal{U}\cap V(z_0)}[n]$. As the support of this last complex of sheaves is contained in $V(z_0)$, if we apply $\psi_{\hat{z}_0}[-1]$, we get the zero complex. Hence, we find that the first complex in (†)

$$\phi_{\hat{z}_0}[-1]\phi_f[-1]k_*k^*\mathbf{P}^\bullet[-1] \cong \phi_{f_0}[-1]\mathbb{Z}^\bullet_{\mathcal{U}\cap V(z_0)}[n].$$

Note that this is a perverse sheaf with isolated support at the origin.

2) As $\mathbf{P}^\bullet$ is perverse and l is the inclusion of a hypersurface complement, $l_!l^!\mathbf{P}^\bullet$ is perverse. Thus, the second complex of (†), $\phi_{\hat{z}_0}[-1]\phi_f[-1]l_!l^!\mathbf{P}^\bullet$, is perverse.

3) As $\mathbf{P}^\bullet$ is perverse and the origin is an isolated point in the intersection of $V(z_0)$ and Σf, $\phi_{\hat{z}_0}[-1]\phi_f[-1]\mathbf{P}^\bullet$ is perverse and has the origin as an isolated point in its support.

At isolated points in their supports, perverse sheaves have their stalk cohomology concentrated in degree zero. Thus, the long-exact sequence on the stalk cohomology at the origin obtained from (†) has at most one non-trivial piece – namely, the short exact sequence

$$(\ddagger) \quad 0 \rightarrow H^0\big(\phi_{f_0}[-1]\mathbb{Z}^\bullet_{\mathcal{U}\cap V(z_0)}[n]\big)_{\mathbf{0}} \rightarrow H^0\big(\phi_{\hat{z}_0}[-1]\phi_f[-1]l_!l^!\mathbf{P}^\bullet\big)_{\mathbf{0}} \rightarrow H^0\big(\phi_{\hat{z}_0}[-1]\phi_f[-1]\mathbf{P}^\bullet\big)_{\mathbf{0}} \rightarrow 0.$$

Observe that $H^0\big(\phi_{f_0}[-1]\mathbb{Z}^\bullet_{\mathcal{U}\cap V(z_0)}[n]\big)_{\mathbf{0}} \cong \mathbb{Z}^{\mu_0}$ and that, by [**7**] or [**9**],

$$H^0\big(\phi_{\hat{z}_0}[-1]\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]\big)_{\mathbf{0}} \cong \mathbb{Z}^{\lambda^0_f}.$$

Therefore, we define the nexus N to be $H^0\big(\phi_{\hat{z}_0}[-1]\phi_f[-1]l_!l^!\mathbf{P}^\bullet\big)_{\mathbf{0}}$, and (‡) becomes the horizontal row of the nexus diagram.

To obtain the vertical row of the nexus diagram, we begin by considering another fundamental distinguished triangle

$$\hat{k}^*[-1]\phi_f[-1]l_!l^!\mathbf{P}^\bullet \to \psi_{\hat{z}_0}[-1]\phi_f[-1]l_!l^!\mathbf{P}^\bullet \to \phi_{\hat{z}_0}[-1]\phi_f[-1]l_!l^!\mathbf{P}^\bullet \xrightarrow{[1]}$$

and turn this triangle to obtain

$$\psi_{\hat{z}_0}[-1]\phi_f[-1]l_!l^!\mathbf{P}^\bullet \to \phi_{\hat{z}_0}[-1]\phi_f[-1]l_!l^!\mathbf{P}^\bullet \to \hat{k}^*\phi_f[-1]l_!l^!\mathbf{P}^\bullet \xrightarrow{[1]} .$$

Again, we look at the associated long exact sequence on the stalk cohomology at the origin.

4) As $l_!l^!\mathbf{P}^\bullet$ is isomorphic to $\mathbf{P}^\bullet$ outside of $V(z_0)$, it follows that

$$\psi_{\hat{z}_0}[-1]\phi_f[-1]l_!l^!\mathbf{P}^\bullet \ \cong\ \psi_{\hat{z}_0}[-1]\phi_f[-1]\mathbf{P}^\bullet.$$

Since $\mathbf{P}^\bullet$ is perverse and the origin is an isolated point in the intersection of $V(z_0)$ and Σf, $\psi_{\hat{z}_0}[-1]\phi_f[-1]\mathbf{P}^\bullet$ is perverse and has the origin as an isolated point in its support. Thus,

$$H^0\big(\psi_{\hat{z}_0}[-1]\phi_f[-1]l_!l^!\mathbf{P}^\bullet\big)_{\mathbf{0}} \ \cong\ H^0\big(\psi_{\hat{z}_0}[-1]\phi_f[-1]\mathbf{P}^\bullet\big)_{\mathbf{0}} \ \cong\ \bigoplus_i \mathbb{Z}^{\mu_i}.$$

5) Note that $H^*\big(\hat{k}^*\phi_f[-1]l_!l^!\mathbf{P}^\bullet)_0 \ \cong\ H^*\big(\phi_f[-1]l_!l^!\mathbf{P}^\bullet)_0$ As the nexus is defined to be $N = H^0\big(\phi_{\hat{z}_0}[-1]\phi_f[-1]l_!l^!\mathbf{P}^\bullet\big)_{\mathbf{0}}$, we would be finished if we could show that $H^*\big(\phi_f[-1]l_!l^!\mathbf{P}^\bullet)_0$ is zero outside of degree zero and that

$$H^0\big(\phi_f[-1]l_!l^!\mathbf{P}^\bullet)_0 \cong \mathbb{Z}^\omega.$$

Consider the distinguished triangle

$$\big(l_!l^!\mathbf{P}^\bullet\big)_{|_{V(f)}} \to \psi_f[-1]l_!l^!\mathbf{P}^\bullet \to \phi_f[-1]l_!l^!\mathbf{P}^\bullet \xrightarrow{[1]} .$$

As $l_!$ is the extension by zero, we find that

$$H^0\big(\phi_f[-1]l_!l^!\mathbf{P}^\bullet)_0 \cong H^0\big(\psi_f[-1]l_!l^!\mathbf{P}^\bullet)_0.$$

Now, consider the distinguished triangle

$$\psi_f[-1]l_!l^!\mathbf{P}^\bullet \to \psi_f[-1]\mathbf{P}^\bullet \xrightarrow{\xi} \psi_f[-1]k_*k^*\mathbf{P}^\bullet \xrightarrow{[1]} .$$

The stalk at $\mathbf{0}$ of the map ξ is the map induced by Lê's attaching map in [**3**]; this is Morse-theoretic map involved in Lê's description of how the Milnor fiber of f is obtained from the Milnor fiber of f_0. Therefore, $H^*\big(\psi_f[-1]l_!l^!\mathbf{P}^\bullet)_0$ is isomorphic to the relative cohomology $H^*(F, F_0;\ \mathbb{Z})$, and Lê's result tells us that $H^i\big(\psi_f[-1]l_!l^!\mathbf{P}^\bullet)_0 = 0$, unless $i = 0$, and

$$H^0\big(\psi_f[-1]l_!l^!\mathbf{P}^\bullet)_0 \cong \mathbb{Z}^\omega.$$

This is what we needed to prove.

§3. **Applications**.

Simplification/Application 1. We will use the Main Theorem to simplify the non-splitting result of Lê which appears in [**4**]. The heart of the argument is the same; it is the details and "trick" at the end of Lê's proof that we can eliminate.

Suppose that $\mu_0 = \lambda^1_f$, i.e., that $\mu_0 = \sum_i \mu_i$. We wish to conclude that there is exactly one point in $H_t \cap \Sigma f$, i.e., that the intersection number of reduced varieties $\big(|\Sigma f| \cdot V(z_0)\big)_{\mathbf{0}}$ equals one. This is equivalent to saying that Σf has a single smooth component which is transversely intersected by $V(z_0)$.

As $\mu_0 = \lambda^1_f$, from the nexus diagram we conclude that $\omega = \lambda^0_f$. However, as discussed in the Introduction, this implies that $\omega = \lambda^0_f = 0$. Thus, α and γ are isomorphisms, and $\alpha^{-1} \circ \gamma$ yields an isomorphism from $\widetilde{H}^{n-1}(F_0;\ \mathbb{Z})$ to $\bigoplus_i \widetilde{H}^{n-1}(F_i;\ \mathbb{Z})$. As the f-monodromy maps act commutatively on the nexus diagram, $(\alpha^{-1} \circ \gamma) \circ h_0 = (\oplus_i h_i) \circ (\alpha^{-1} \circ \gamma)$. Thus the trace of h_0 equals the sum of the traces of the h_i's, and applying the result of A'Campo [**1**] on the Lefschetz number of the monodromy, we find

$$(-1)^n = \big(|\Sigma f| \cdot V(z_0)\big)_{\mathbf{0}}(-1)^n,$$

and hence $\big(|\Sigma f| \cdot V(z_0)\big)_{\mathbf{0}} = 1$.

Note that our proof that $\big(|\Sigma f| \cdot V(z_0)\big)_{\mathbf{0}} = 1$ is precisely the argument given by Lê in [**4**], **except** that we do not have to use a Mayer-Vietoris sequence and we do not have to perform a "trick" to eliminate the eigenvalue 1 from the monodromy.

Also note that the nexus diagram also implies a well-known result from [**5**]: that, in the present case, $\widetilde{H}^n(F;\ \mathbb{Z}) = 0$ and $\widetilde{H}^{n-1}(F;\ \mathbb{Z}) \cong \widetilde{H}^{n-1}(F_0;\ \mathbb{Z})$.

Application 2. The fact that $\widetilde{H}^{n-1}(F;\ \mathbb{Z}) \cong \ker(\delta \circ \alpha)$ immediately implies that the rank, $\tilde{b}_{n-1}$, of $\widetilde{H}^{n-1}(F;\ \mathbb{Z})$ is at most λ^1_f. Assume that $\tilde{b}_{n-1} = \lambda^1_f$; we wish to see what this implies.

Recall from the Introduction that $\tilde{b}_{n-1} \leqslant \sum_\nu \mu_\nu$. As $\lambda^1_f = \sum_\nu (\nu \cdot V(z_0))_{\mathbf{0}}\, \mu_\nu$, we see that we must have that, for every component ν of Σf, $(\nu \cdot V(z_0))_{\mathbf{0}} = 1$, i.e., each ν is smooth at the origin and is transversely intersected by $V(z_0)$. Thus, $\big(|\Sigma f| \cdot V(z_0)\big)_{\mathbf{0}}$ simply equals the number of irreducible components of Σf through the origin.

Now, the Main Theorem implies that $\widetilde{H}^{n-1}(F;\ \mathbb{Z})$ injects into $\bigoplus_i \widetilde{H}^{n-1}(F_i;\ \mathbb{Z}) \cong \oplus_i \mathbb{Z}^{\mu_i} \cong \mathbb{Z}^{\lambda^1_f}$ by an injection which commutes with the respective monodromies. Therefore, since we are assuming that $\widetilde{H}^{n-1}(F;\ \mathbb{Z}) \cong \mathbb{Z}^{\lambda^1}$, we conclude that the characteristic polynomials of the f-monodromy on $\widetilde{H}^{n-1}(F;\ \mathbb{Z})$ and on $\bigoplus_i \widetilde{H}^{n-1}(F_i;\ \mathbb{Z})$ must be equal, i.e., $\mathrm{char}_{m_{n-1}}(t) = \prod_i \mathrm{char}_{h_i}(t)$.

Consider a portion of the long exact cohomology sequence of the pair (F, F_0):

$$\cdots \to 0 \to \widetilde{H}^{n-1}(F;\ \mathbb{Z}) \xrightarrow{\rho} \widetilde{H}^{n-1}(F_0;\ \mathbb{Z}) \xrightarrow{\zeta} H^n(F, F_0;\ \mathbb{Z}) \to \dots,$$

on which the f-monodromy acts commutatively. The f-monodromy induces an automorphism on the image of ζ; let $p : \operatorname{im}\zeta \to \operatorname{im}\zeta$ denote this automorphism. As $H^n(F, F_0;\ \mathbb{Z}) \cong \mathbb{Z}^\omega$ is free Abelian, $\operatorname{im}\zeta$ is free Abelian of rank $\mu_0 - \lambda^1_f$, and so $\mathrm{char}_{h_0}(t) = \mathrm{char}_{m_{n-1}}(t) \cdot \mathrm{char}_p(t)$.

Therefore, from our previous two paragraphs, we conclude that

$$(\dagger) \qquad \mathrm{char}_{h_0}(t) = \mathrm{char}_p(t) \cdot \prod_i \mathrm{char}_{h_i}(t),$$

where each characteristic polynomial is in $\mathbb{Z}[t]$ and the degree of $\mathrm{char}_p(t)$ is $\mu_0 - \lambda^1_f$. Furthermore, as all of the complex eigenvalues of h_0 are roots of unity, we see that the complex eigenvalues of p are also roots of unity. Finally, applying A'Campo's Lefschetz number result to h_0 and to each h_i, we see that the trace of p equals $(-1)^{n+1}\Big(\big(|\Sigma f|\cdot V(z_0)\big)_{\mathbf{0}} - 1\Big)$.

As each root of $\mathrm{char}_p(t)$ has complex norm equal to 1, we conclude immediately that

$$(\ddagger)\qquad \big(|\Sigma f|\cdot V(z_0)\big)_{\mathbf{0}} - 1 \leqslant \mu_0 - \lambda^1_f$$

with equality implying that all of the eigenvalues of p are equal to $(-1)^{n+1}$ and, hence, that h_0 has (at least) $\mu_0 - \lambda^1_f$ eigenvalues equal to $(-1)^{n+1}$.

• The case in Simplification/Application 1 was that of $\tilde{b}_{n-1} = \lambda^1_f$ and $\mu_0 - \lambda^1_f = 0$. From ($\ddagger$), we recover that $\big(|\Sigma f|\cdot V(z_0)\big)_{\mathbf{0}} = 1$ in this case.

• Consider now the next easiest case where $\tilde{b}_{n-1} = \lambda^1_f$ and $\mu_0 - \lambda^1_f = 1$. Then, ($\dagger$), and the paragraph beneath it, implies that $\mathrm{char}_p(t) = t - (-1)^{n+1}$ and $\big(|\Sigma f|\cdot V(z_0)\big)_{\mathbf{0}} = 2$. This implies that Σf has two smooth components which are each transversely intersected by $V(z_0)$, and that $(-1)^{n+1}$ is an eigenvalue of h_0.

Thus, we recover a slightly-improved version of the result of [**6**]: if $\mu_0 - \lambda^1_f = 1$, then either Σf consists of two smooth components which are transversely intersected by $V(z_0)$ at the origin (e.g., $f = (z_0^2 - z_1^2 - z_2)z_2$), or the rank of $\widetilde{H}^{n-1}(F;\ \mathbb{Z})$ is strictly less than λ^1_f.

• The cases get progressively more complicated as $\mu_0 - \lambda^1_f$ increases. Consider the case where $\tilde{b}_{n-1} = \lambda^1_f$ and $\mu_0 - \lambda^1_f = 2$. The reader is invited to show that ($\dagger$), and the paragraph beneath it, implies that there are three possible cases:

i) Σf has one component and $\mathrm{char}_p(t) = t^2 \pm 1$;

ii) Σf has two components and $\mathrm{char}_p(t) = t^2 - (-1)^{n+1}t + 1$;

iii) Σf has three components and $\mathrm{char}_p(t) = \big(t - (-1)^{n+1}\big)^2$.

In each case, each component of Σf is smooth and transversely intersected by $V(z_0)$ at $\mathbf{0}$.

Application 3. Suppose that f_0 is homogeneous of degree d_0. By using Theorem 9.6 of [**10**], or directly from [**11**], the characteristic polynomial of the f_0-monodromy is given by

$$(\dagger)\qquad \mathrm{char}_{h_0}(t) = (t-1)^{a_0}\left(\frac{t^{d_0}-1}{t-1}\right)^{b_0},$$

where $b_0 := \dfrac{(d_0-1)^n - (-1)^n}{d_0}$ and $a_0 := b_0 + (-1)^n = (d_0-1)\left(\dfrac{(d_0-1)^{n-1} - (-1)^{n-1}}{d_0}\right)$.

Let Φ_k denote the k-th cyclotomic polynomial, i.e., $\Phi_k = \prod_\xi (t-\xi)$ where the ξ vary over the primitive k-th roots of unity. Then, the main theorem immediately tells us that

$$\mathrm{char}_{m_{n-1}}(t) = \prod_{k|d_0} \Phi_k^{c_k},$$

where $c_1 \leqslant a_0$ and, for $k>1$, $c_k \leqslant b_0$; in addition, for all $k|d_0$, $\Phi_k^{c_k}$ divides $\prod_\nu \mathrm{char}_{h_\nu}(t)$ in $\mathbb{Z}[t]$.

This is particularly useful in the special case where, at each point $\mathbf{p}_i \in H_t \cap \Sigma f$, f_t is "homogeneous at $\mathbf{p}_i$", i.e., $g_i(\mathbf{z}) := f_t(\mathbf{z}+\mathbf{p}_i)$ is homogeneous. Let d_i denote the degree of g_i. For all $\mathbf{p}_i$ in a given component ν of Σf, the d_i must be the same; denote this common value by d_ν. For each ν, there is a formula analogous to (†) for the characteristic polynomial of h_ν, and so the irreducible factors of $\prod_\nu \mathrm{char}_{h_\nu}(t)$ are cyclotomic polynomials Φ_k for which k must divide one of the d_ν.

An example of such an f would be one which defines a central arrangement of d_0 hyperplanes in $\mathbb{C}^3$. The above paragraphs generalize most of the main theorem from [**8**], where we considered only hyperplane arrangements in $\mathbb{C}^3$. This is related to the work of [**2**].

One should also compare this application to Siersma's zeta function formula [**12**, p. 125] for homogeneous functions with 1-dimensional critical loci.

§4. **Further Remarks**.

Regardless of the dimension of the critical locus of f, a nexus diagram exists in the category of perverse sheaves.

In the proof of the Main Theorem, even before we took the stalk cohomology at the origin, we had two short exact sequences in the category of perverse sheaves; that is, we had a perverse nexus diagram. However, since each of these perverse sheaves had the origin as an isolated point in their support, taking stalk cohomology essentially did nothing.

If the dimension of Σf is arbitrary and z_0 is generic, we still obtain a perverse nexus diagram

$$\begin{array}{ccccc} & & \psi_{\hat{z}_0}[-1]\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] & & \\ & & \downarrow{\scriptstyle \alpha} & & \\ \phi_{f_0}[-1]\mathbb{Z}^\bullet_{\mathcal{U}\cap V(z_0)}[n] & \xrightarrow{\gamma} & \mathbf{N}^\bullet & \xrightarrow{\delta} & j_*\mathbb{Z}_{\mathbf{0}}^{\lambda_f^0} \\ & & \downarrow{\scriptstyle \beta} & & \\ & & j_*\mathbb{Z}^{\omega}_{\mathbf{0}}, & & \end{array}$$

where j denotes the inclusion of the origin into $V(f, z_0)$. The f and z_0 monodromies act commutatively on this diagram.

We write $\ker^p$ and coker^p for the kernel and cokernel in the category of perverse sheaves, and we write ${}^pH^*$ for the perverse cohomology. Then, in general, in the perverse nexus diagram above,

$$\ker^p(\beta\circ\gamma) \;\cong\; \ker^p(\delta\circ\alpha) \;\cong\; {}^pH^{-1}\big(\hat{k}^*\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]\big)$$

and

$$\mathrm{coker}^p(\beta\circ\gamma) \;\cong\; \mathrm{coker}^p(\delta\circ\alpha) \;\cong\; {}^pH^{0}\big(\hat{k}^*\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]\big).$$

Furthermore, there is an analogous perverse nexus diagram involving $\psi_f[-1]$, in place of $\phi_f[-1]$. That is, there is a nexus diagram in the category of perverse sheaves obtained from the above diagram by replacing each occurrence of $\phi_f[-1]$ by $\psi_f[-1]$, and by replacing λ_f^0 by ω. The perverse kernel and cokernel statements also hold with $\phi_f[-1]$ replaced by $\psi_f[-1]$.

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