SINGULARITIES AND ENRICHED CYCLES

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Abstract. We introduce graded, enriched characteristic cycles as a method for encoding Morse modules of strata with respect to a constructible complex of sheaves. Using this new device, we obtain results for arbitrary complex analytic functions on arbitrarily singular complex analytic spaces.

§1. Introduction

Let X be a complex analytic space, with arbitrary singularities, and let $\mathbf{p} \in X$. As we are interested in local questions near $\mathbf{p}$, we may assume that X is a (closed) complex analytic subspace of a connected, open subset $\mathcal{U} \subseteq \mathbb{C}^{n+1}$, and $\mathbf{z} := (z_0, \dots, z_n)$ are coordinates on $\mathbb{C}^{n+1}$.

Let $\tilde{f} : \mathcal{U} \to \mathbb{C}$ be a complex analytic function, and let $f := \tilde{f}_{|X}$. Let $\mathcal{S} := \{S_\alpha\}$ be a complex analytic Whitney stratification of X, with connected strata. Fix a base ring R that is a regular, Noetherian ring with finite Krull dimension (e.g., $\mathbb{Z}, \mathbb{Q}$, or $\mathbb{C}$). Let $\mathbf{F}^\bullet$ be a bounded complex of sheaves of R-modules on X, which is constructible with respect to $\{S_\alpha\}$.

In this extremely general situation, we are interested in producing effectively calculable algebraic data which provides information about the topology and geometry of the hypersurface $V(f)$. In particular, we are interested in Thom's a_f condition, and Milnor fibration data. Thus, we are led to study the nearby and vanishing cycles of $\mathbf{F}^\bullet$ along f; that is, respectively, $\psi_f \mathbf{F}^\bullet$ and $\phi_f \mathbf{F}^\bullet$.

Our method for producing algebraic data is motivated by our results on Lê cycles and Lê numbers ([**M3**], [**M4**], [**M5**]), which corresponds to the case of an affine hypersurface where $\mathbf{F}^\bullet$ is the constant sheaf $\mathbb{C}^\bullet_{\mathcal{U}}$. Much of our past work has centered around the problems of extended our work on Lê cycles to the general setting of this paper. In [**M8**], we associated cycles to arbitrary complexes of sheaves. In [**M1**], [**M2**], we studied the special cases of $\psi_f \mathbf{F}^\bullet$ and $\phi_f \mathbf{F}^\bullet$. In [**M7**], we gave a somewhat satisfactory generalization; we defined *Lê-Vogel cycles and numbers.*

The Lê cycles of [**M7**] were produced by starting with the characteristic cycle of the complex $\mathbf{F}^\bullet$, and then using a Lê-Vogel process on an ideal, J, defining the image of $d\tilde{f}$ in $T^*\mathcal{U}$. It was crucial that we had a result which told us how the characteristic cycle of $\phi_f \mathbf{F}^\bullet$ is obtained from the characteristic cycle of $\mathbf{F}^\bullet$ via blowing-up J; we refer to this result as the *vanishing index theorem.*

The fundamental weakness of the results of [**M7**] lies in the fact that the characteristic cycle of $\mathbf{F}^\bullet$ contains only Euler characteristic data on the Morse modules to strata, and thus disposes of a great deal of cohomological data which describes the structure of $\mathbf{F}^\bullet$. In [**M7**] and [**M9**], we used the functorial properties of perverse cohomology to extract more refined data from characteristic cycles. In [**M6**], we showed with some difficulty how the microlocal theory of Kashiwara and Schapira could be used to obtain better results in the special case of an isolated critical point.

In this paper, we show how our results from [**M7**] can easily be turned into stronger results – without using the devices of perverse cohomology or microlocal theory – simply by using cycles with module-coefficients; these are our *enriched cycles.* Since we wish for our enriched cycles to

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be able to encode the cohomology modules of normal data to strata (with coefficients in $\mathbf{F}^\bullet$), we consider graded, enriched cycles and define $\mathrm{gecc}^\bullet(\mathbf{F}^\bullet)$, the *graded, enriched, characteristic cycle of* $\mathbf{F}^\bullet$. The fact that $\mathrm{gecc}^0\left({}^\mu H^k(\mathbf{F}^\bullet)\right) = \mathrm{gecc}^k(\mathbf{F}^\bullet)$ explains why we may use $\mathrm{gecc}^\bullet$ to obtain any of the results of [**M7**] that used the standard characteristic cycle and perverse cohomology.

The fundamental philosophy of this paper is: any result on perverse sheaves which is proved by using characteristic cycles and intersection theory can be generalized to arbitrary bounded, constructible complexes of sheaves by using graded, enriched characteristic cycles and a corresponding enriched intersection theory. Most of this paper consists of demonstrating applications of this general philosophy.

However, there are several important/interesting results in this paper which are not simply enriched forms of earlier results. The equivalent characterizations of *isolating coordinates* given in Theorem 5.10 yield a substantial improvement over [**M7**] and [**M8**] in the level of genericity required of our choice of coordinates when generalizing absolute polar varieties; this is very important since we are trying to produce a method for making effective computations.The result of Theorem 4.8 – necessary and sufficient conditions for Thom's a_f condition – is important in Theorem 6.5 for similar reasons: we use 4.8 and 5.10 to produce a useful notion of "generic coordinates" in the relative situation. Proposition 6.11 tells us that, when the dimension of the critical locus is small, we may check whether our coordinates are generic enough "on-the-fly". Another interesting result which falls out of the enriched approach is a sort of converse to the theorem that: if $\mathbf{F}^\bullet$ is perverse, then $\phi_f\mathbf{F}^\bullet$ is perverse. In Corollary 3.7, we show that: if f is in the square of the maximal ideal of X at $\mathbf{p}$ and $\phi_f\mathbf{F}^\bullet$ is perverse in a neighborhood of $\mathbf{p}$, then $\mathbf{F}^\bullet$ is perverse in a neighborhood of $\mathbf{p}$.

The remaining sections of this paper are organized as follows.

In Section 2, we define enriched cycles, describe a corresponding enriched intersection theory (for proper intersections in smooth manifolds), and define and give some basic properties of the graded, enriched characteristic cycle of a complex of sheaves.

Section 3 contains enriched versions of a number of previous results. Of particular importance is Theorem 3.4, an enriched form of our vanishing index theorem. The only new results in this section are the corollaries to 3.4.

In Section 4, we discuss partitions, stratifications, Thom's a_f condition, and *essentially transverse coordinates*. For most of our results, we do not need the full power of a Whitney stratification, instead we use $\mathbf{F}^\bullet$-partitions. This weakening of the type of "stratification" used leads to stronger results. The genericity that we use on the coordinates in Section 6 will be that the coordinates are essentially transverse to a type of partition.

In Section 5, we look at extreme generalizations of polar varieties and polar multiplicities; we define the *graded, enriched characteristic polar cycles* and the *characteristic polar modules*. We define $\mathbf{F}^\bullet$-*isolating coordinates* and characterize them in terms of intersections with $\mathrm{gecc}^\bullet(\mathbf{F}^\bullet)$. This provides the fundamental relation between enriched intersections with $\mathrm{gecc}^\bullet(\mathbf{F}^\bullet)$ and iterated vanishing and nearby cycles.

In Section 6, we apply the results of Section 5 to the special case of $\phi_f\mathbf{F}^\bullet$. By combining the work of Section 5 with an enriched version of the vanishing index theorem, we obtain our generalization of the Lê cycles and Lê numbers: the *graded, enriched Lê-Vogel cycles and the Lê-Vogel modules.* Our work on essentially transverse coordinates and the a_f condition from Section 4 are used in this section to give sufficient conditions on the choice of coordinates.

In Section 7, we give a sample calculation of Lê-Vogel cycles. We compare and contrast this with other methods of analyzing the perverse cohomology.

Section 8 contains some concluding remarks and questions.

§2. **Enriched Cycles**

We will now define enriched cycles, graded enriched cycles, the graded enriched characteristic cycle, and operations on them. The notions that we define are easy and/or obvious; hence, this section consists of a series of definitions with some remarks.

We continue with all of the notation from the Introduction. We also need to establish some more notation that we shall use throughout the remainder of this paper.

The cotangent bundle $T^*\mathcal{U} \xrightarrow{\eta} \mathcal{U}$ is a trivial bundle; we will frequently use that $T^*\mathcal{U} \cong \mathcal{U} \times \mathbb{C}^{n+1}$ and write simply that η is the projection from $\mathcal{U} \times \mathbb{C}^{n+1}$ onto $\mathcal{U}$. A linear choice of coordinates $\mathbf{z}$ for $\mathcal{U}$ determines a basis $dz_0, \dots, dz_n$ for the cotangent spaces, and we use $\mathbf{w} := (w_0, \dots, w_n)$ for the cotangent coordinates.

Using these coordinates, the image of $d\tilde{f}$ in $T^*\mathcal{U}$, $\operatorname{im} d\tilde{f}$, is given by

$$V\Big(w_0 - \frac{\partial \tilde{f}}{\partial z_0}, \dots, w_n - \frac{\partial \tilde{f}}{\partial z_n}\Big).$$

As $T^*\mathcal{U}$ is (complex) conic, we may projectivize in the cotangent directions, and consider $\mathbb{P}(T^*\mathcal{U}) \cong \mathcal{U} \times \mathbb{P}^n$. In this and the remaining sections of this paper, it will be important for us to consider all four projection maps:

$$\begin{array}{ccc} \mathcal{U} \times \mathbb{C}^{n+1} \times \mathbb{P}^n & \xrightarrow{\pi} & \mathcal{U} \times \mathbb{P}^n \\ \xi \big\downarrow & & \big\downarrow \nu \\ \mathcal{U} \times \mathbb{C}^{n+1} & \xrightarrow{\eta} & \mathcal{U}. \end{array}$$

The conormal bundle to a stratum S_α is given by

$$T^*_{S_\alpha}\mathcal{U} := \big\{(\mathbf{x}, \omega) \in T^*\mathcal{U} \mid \mathbf{x} \in S_\alpha,\ \omega(T_{\mathbf{x}} S_\alpha) \equiv 0\big\}.$$

We are frequently interested in the closure, $\overline{T^*_{S_\alpha}\mathcal{U}}$, of $T^*_{S_\alpha}\mathcal{U}$ inside $T^*\mathcal{U}$. Both $T^*_{S_\alpha}\mathcal{U}$ and $\overline{T^*_{S_\alpha}\mathcal{U}}$ are conic and may be projectivized to yield $\mathbb{P}\big(T^*_{S_\alpha}\mathcal{U}\big)$ and $\mathbb{P}\big(\overline{T^*_{S_\alpha}\mathcal{U}}\big) = \overline{\mathbb{P}\big(T^*_{S_\alpha}\mathcal{U}\big)}$.

Finally, it is fundamental throughout this paper that the functors ψ_f and ϕ_f, shifted by -1, commute, up to natural isomorphism, with (middle perversity) perverse cohomology ${}^{\mu}H^*$; see [**BBD**], 10.3.13 of [**K-S2**] and Remark 6.0.6 of Section 6.0.4 of [**S**] (but, be aware that the definition of the vanishing cycles used in [**K-S2**] is shifted by 1 from the standard definition that we use). In particular, the shifted nearby and vanishing cycles are *perverse functors*, i.e., they take perverse sheaves to perverse sheaves. For the nearby cycles, this was first proved by Goresky and MacPherson in [**G-M1**]. The first proof of which we are aware that the shifted vanishing cycle functor is perverse appears in 1.7 of [**Br**]. For these reasons, we shall always include a shift of -1 when we use the nearby and vanishing cycle functors; we shall write $\psi_f[-1]$ and $\phi_f[-1]$ for the functors ψ_f and ϕ_f composed with a shift by -1.

Definition 2.1. An *enriched cycle*, E, in X is a formal, locally finite sum $\sum_V E_V[V]$, where the V's are irreducible analytic subsets of X and the E_V's are finitely-generated R-modules. We refer to the V's as the *components* of E, and to E_V as the *V-component module of E*. Two enriched cycles are considered the same provided that all of the component modules are isomorphic. The underlying set of E is $|E| := \cup_{E_V \neq 0} V$.

If $C = \sum n_V[V]$ is an ordinary positive cycle in X, i.e., all of the n_v are non-negative integers, then there is a corresponding enriched cycle $[C]^{\mathrm{enr}}$ in which the V-component module is the free R-module of rank n_V. If R is an integral domain, so that rank of an R-module is well-defined, then an enriched cycle E yields an ordinary cycle $[E]^{\mathrm{ord}} := \sum_V (\mathrm{rk}(E_V))[V]$.

If q is a finitely-generated module and E is an enriched cycle, then we let $qE := \sum_V (q \otimes E_V)[V]$; thus, if R is an integral domain and E is an enriched cycle, $[qE]^{\mathrm{ord}} = (\mathrm{rk}(q))[E]^{\mathrm{ord}}$ and if C is an ordinary positive cycle and n is a positive integer, then $[nC]^{\mathrm{enr}} = R^n[C]^{\mathrm{enr}}$.

The (direct) sum of two enriched cycles D and E is given by $(D+E)_V := D_V \oplus E_V$.

There is a partial ordering on enriched cycles given by: $D \leqslant E$ if and only if there exists an enriched cycle P such that $D + P = E$. This relation is clearly reflexive and transitive; moreover, anti-symmetry follows from the fact that if M and N are Noetherian modules such that $M \oplus N \cong M$, then $N = 0$.

If two irreducible analytic subsets V and W intersect properly in $\mathcal{U}$, then the (ordinary) intersection cycle $[V] \cdot [W]$ is a well-defined positive cycle; we define the enriched intersection product of $[V]^{\mathrm{enr}}$ and $[W]^{\mathrm{enr}}$ by $[V]^{\mathrm{enr}} \odot [W]^{\mathrm{enr}} = ([V] \cdot [W])^{\mathrm{enr}}$. If D and E are enriched cycles, and every component of D properly intersects every component of E in $\mathcal{U}$, then we say that *D and E intersect properly* in $\mathcal{U}$ and we extend the intersection product linearly, i.e., if $D = \sum_V D_V[V]$ and $E = \sum_W E_W[W]$, then

$$D \odot E := \sum_{V,W} (D_V \otimes E_W)([V] \cdot [W])^{\mathrm{enr}}.$$

Suppose that $\mathbf{g} := (g_0, \dots, g_d)$ is a $(d+1)$-tuple of analytic functions on $\mathcal{U}$ and the $D = \sum D_V[V]$ is an enriched cycle in $\mathcal{U}$. Then, for each V, the blow-up of the ideal generated by the restriction of $\mathbf{g}$ to V, $\mathrm{Bl}_{\mathbf{g}}(V)$, yields an irreducible analytic subset of $\mathcal{U} \times \mathbb{P}^d$; in addition, we obtain an exceptional divisor $\mathrm{Ex}_{\mathbf{g}}(V)$, which is an (ordinary) positive cycle in $\mathcal{U} \times \mathbb{P}^d$. We define the *blow-up and exceptional divisor of D along $\mathbf{g}$* to be the enriched cycles in $\mathcal{U} \times \mathbb{P}^d$ given by $\mathrm{Bl}_{\mathbf{g}}(D) := \sum_V D_V[\mathrm{Bl}_{\mathbf{g}}(V)]^{\mathrm{enr}}$ and $\mathrm{Ex}_{\mathbf{g}}(D) := \sum_V D_V[\mathrm{Ex}_{\mathbf{g}}(V)]^{\mathrm{enr}}$, respectively. If we had started with an ordinary cycle D, we would analogously obtained ordinary cycles for $\mathrm{Bl}_{\mathbf{g}}(D)$ and $\mathrm{Ex}_{\mathbf{g}}(D)$.

A *graded, enriched cycle* $E^\bullet$ is simply an enriched cycle E^i for i in some bounded set of integers. An single enriched cycle is considered as a graded enriched cycle by being placed totally in degree zero. The analytic set V is a *component of* $E^\bullet$ if and only if V is a component of E^i for some i, and the underlying set of $E^\bullet$ is $|E^\bullet| = \cup_i |E^i|$. If k is an integer, we define the *k-shifted graded, enriched cycle* $E^\bullet[k]$ by $(E^\bullet[k])^i := E^{i+k}$.

If R is a domain, then $E^\bullet$ yields an ordinary cycle

$$[E^\bullet]^{\mathrm{ord}} := \sum_i (-1)^i [E^i]^{\mathrm{ord}} \ = \ \sum_{i,V} (-1)^i (\mathrm{rk}(E^i_V))[V].$$

If q is a finitely-generated module and $E^\bullet$ is a graded enriched cycle, then we define the graded enriched cycle $qE^\bullet$ by $(qE^\bullet)^i := \sum_V (q \otimes E^i_V)[V]$. The (direct) sum of two graded enriched cycles $D^\bullet$ and $E^\bullet$ is given by $(D^\bullet + E^\bullet)^i_V := D^i_V \oplus E^i_V$. If D^i properly intersects E^j for all i and j, then we say that $D^\bullet$ and $E^\bullet$ *intersect properly* and we define the intersection product by

$$(D^\bullet \odot E^\bullet)^k := \sum_{i+j=k} (D^i \odot E^j).$$

Let $\tau : W \to Y$ be a proper morphism between analytic spaces. If $C = \sum n_V[V]$ is an ordinary positive cycle in W, then the proper push-forward $\tau_*(C) = \sum n_V \tau_*([V])$ is a well-defined ordinary cycle. If $E^\bullet = \sum_V E^\bullet_V[V]$ is an enriched cycle in W, then we define the *proper push-forward of* $E^\bullet$ *by* τ to be the graded enriched cycle $\tau^\bullet_*(E^\bullet)$ defined by

$$\tau^j_*(E^\bullet) \ := \ \sum_V E^j_V[\tau_*([V])]^{\mathrm{enr}}.$$

The ordinary projection formula for divisors ([**F**], 2.3.c) immediately implies the following enriched version. Let $E^\bullet$ be a graded enriched cycle in X. Let $W := |E^\bullet|$. Let $\tau : W \to Y$ be a proper morphism, and let $g : Y \to \mathbb{C}$ be an analytic function such that $g \circ \tau$ is not identically zero on any component of $E^\bullet$. Then, g is not identically zero on any component of $\tau^\bullet_*(E^\bullet)$ and

$$\tau^\bullet_*\big(E^\bullet \ \odot \ V(g\circ\tau)\big) \ = \ \tau^\bullet_*(E^\bullet) \ \odot \ V(g).$$

The standard intersection result on conservation of number generalizes easily to:

conservation of module:

Let E be a purely k-dimensional enriched cycle in $\mathcal{U}$ and let $\mathbf{f} := (f_1, \dots, f_k) \in (\mathcal{O}_\mathcal{U})^k$ be such that E and $V(\mathbf{f})$ intersect properly in the isolated point $\mathbf{p}$.

Let $g_1(\mathbf{z}, t), \dots, g_k(\mathbf{z}, t) \in \mathcal{O}_{\mathcal{U}\times\overset{\circ}{\mathbb{D}}}$ be such that $g_i(\mathbf{z}, 0) = f_i(\mathbf{z})$ for all i. For $t_0 \in \overset{\circ}{\mathbb{D}}$, let C_{t_0} be the cycle in $\mathcal{U}$ given by $\big[V(g_1(\mathbf{z}, t_0), \dots, g_k(\mathbf{z}, t_0))\big]$. Then,

$$(E \odot V(\mathbf{f}))_\mathbf{p} \cong \sum_{\mathbf{q}\in \overset{\circ}{B}_\epsilon \cap |E| \cap |C_{t_0}|} \big(E \odot C_{t_0}\big)_\mathbf{q},$$

where $\epsilon > 0$ is sufficiently small, $\overset{\circ}{B}_\epsilon$ is an open ball of radius ϵ centered at $\mathbf{p}$, and $|t_0| \ll \epsilon$.

Definition 2.2. If $D^\bullet$ is a graded, enriched cycle and $\mathbf{g} := (g_0, \dots, g_d)$ is a $(d+1)$-tuple of analytic functions on $\mathcal{U}$, then we define the *blow-up and exceptional divisor of* $D^\bullet$ *along* $\mathbf{g}$ to be the graded, enriched cycles in $\mathcal{U} \times \mathbb{P}^d$ given by $(\mathrm{Bl}_\mathbf{g}(D^\bullet))^i := \sum_V D^i_V[\mathrm{Bl}_\mathbf{g}(V)]^{\mathrm{enr}}$ and $(\mathrm{Ex}_\mathbf{g}(D^\bullet))^i := \sum_V D^i_V[\mathrm{Ex}_\mathbf{g}(V)]^{\mathrm{enr}}$, respectively.

Later, we will be especially interested in the case where $\mathbf{g}$ is the tuple defining $\operatorname{im} d\tilde{f}$,

$$\Big(w_0 - \frac{\partial \tilde{f}}{\partial z_0}, \dots, w_n - \frac{\partial \tilde{f}}{\partial z_n}\Big),$$

and $D^\bullet$ is such that $|D^\bullet|$ is a union of components of the form $\overline{T^*_{S_\alpha}\mathcal{U}}$. In this case, we will denote the blow-up and exceptional divisor by $\mathrm{Bl}_{\operatorname{im} d\tilde{f}}(D^\bullet)$ and $\mathrm{Ex}_{\operatorname{im} d\tilde{f}}(D^\bullet)$, respectively.

Definition 2.3. Suppose that $\mathbf{F}^\bullet$ is a bounded complex of sheaves, which is constructible with respect to an analytic Whitney stratification $\{S_\alpha\}$, in which the strata are connected. Let $d_\alpha := \dim S_\alpha$. If $(\mathbb{N}_\alpha, \mathbb{L}_\alpha)$ is a pair consisting of a normal slice and complex link (see [**G-M2**]), respectively, to the stratum S_α, then, for each integer k, the isomorphism-type of the module $\mathbb{H}^{k-d_\alpha}(\mathbb{N}_\alpha, \mathbb{L}_\alpha; \mathbf{F}^\bullet)$ is independent of the choice of $(\mathbb{N}_\alpha, \mathbb{L}_\alpha)$; we refer to $\mathbb{H}^{k-d_\alpha}(\mathbb{N}_\alpha, \mathbb{L}_\alpha; \mathbf{F}^\bullet)$ as the *degree k Morse module of S_α with respect to* $\mathbf{F}^\bullet$. The *graded, enriched characteristic cycle of* $\mathbf{F}^\bullet$ *in the cotangent bundle* $T^*\mathcal{U}$ is defined in degree k to be

$$\operatorname{gecc}^k(\mathbf{F}^\bullet) := \sum_\alpha H^{k-d_\alpha}(\mathbb{N}_\alpha, \mathbb{L}_\alpha; \mathbf{F}^\bullet)\big[\,\overline{T^*_{S_\alpha}\mathcal{U}}\,\big].$$

Remark 2.4. There are no canonical choices for defining the the normal slices or complex links of strata (again, see [**G-M2**]). However, as two enriched cycles are equal provided that the component modules are all isomorphic, the graded, enriched characteristic cycle is well-defined.

Note that the ordinary characteristic cycle, $\operatorname{Ch}(\mathbf{F}^\bullet)$, of $\mathbf{F}^\bullet$ is related to the graded, enriched characteristic cycle of $\mathbf{F}^\bullet$ by $\operatorname{Ch}(\mathbf{F}^\bullet) = (-1)^{\dim X}[\operatorname{gecc}^\bullet(\mathbf{F}^\bullet)]^{\mathrm{ord}}$, provided that the base ring is an integral domain.

On a different note, recall that a complex $\mathbf{F}^\bullet$ is called *pure* (of shift 0) provided that $\operatorname{gecc}^\bullet(\mathbf{F}^\bullet)$ is concentrated in degree 0 (see [**K-S1**], 7.2 and 9.5). If $\mathbf{P}^\bullet$ is a perverse sheaf on X, then $\mathbf{P}^\bullet$ is pure; the converse of this is also true (see 9.5.2 of [**K-S1**] or Remark 2.6).

Finally, note that, for all k, $\operatorname{gecc}^k(\mathbf{F}^\bullet[i]) = (\operatorname{gecc}^k(\mathbf{F}^\bullet))[i]$.

Proposition 2.5. *For all k, there is an equality of enriched cycles given by*

$$\operatorname{gecc}^0\left({}^\mu H^k(\mathbf{F}^\bullet)\right) = \operatorname{gecc}^k(\mathbf{F}^\bullet).$$

Furthermore, there are equalities of sets given by $\operatorname{supp}{}^\mu H^k(\mathbf{F}^\bullet) = \eta(|\operatorname{gecc}^k(\mathbf{F}^\bullet)|)$, *and* $\operatorname{supp}(\mathbf{F}^\bullet) = \eta(|\operatorname{gecc}^\bullet(\mathbf{F}^\bullet)|)$.

Proof. The first equality follows immediately from the fact that perverse cohomology commutes with taking the shifted vanishing cycles; the argument is precisely the same as that of Proposition 2.1 in [**M9**].

If $\mathbf{P}^\bullet$ is a perverse sheaf, then $\operatorname{supp}(\mathbf{P}^\bullet) = \eta(|\operatorname{gecc}^0(\mathbf{P}^\bullet)|)$; this is essentially the last part of Lemma 3.1 of [**M1**], and the proof is the same. The second equality of the proposition follows by applying this to ${}^\mu H^k(\mathbf{F}^\bullet)$, and then using the first equality.

As $\operatorname{supp}(\mathbf{F}^\bullet) = \bigcup_k \operatorname{supp}{}^\mu H^k(\mathbf{F}^\bullet)$ (see [**K-S2**]), the last equality of the proposition follows from the second.

Alternatively, the last two equalities could be concluded from the fact that, at a generic point in an irreducible component of $\operatorname{supp}(\mathbf{F}^\bullet)$, the Morse modules simply yield stalk cohomology. □

Remark 2.6. The equality of $\operatorname{supp}{}^\mu H^k(\mathbf{F}^\bullet)$ and $\eta(|\operatorname{gecc}^k(\mathbf{F}^\bullet)|)$ implies that pure complexes have ${}^\mu H^k(\mathbf{F}^\bullet) = 0$ for $k \neq 0$, i.e., we recover the result of 9.5.2 of [**K-S1**] that pure complexes are perverse.

We should also remark here that the equality of $\operatorname{supp}(\mathbf{F}^\bullet)$ and $\eta(|\operatorname{gecc}^\bullet(\mathbf{F}^\bullet)|)$ explains why a number of our later results have hypotheses involving $|\operatorname{gecc}^\bullet(\mathbf{F}^\bullet)|$ even though we wish to conclude results in individual degrees.

Definition 2.7. An irreducible subvariety Y of X is an *essential subvariety for* $\mathbf{F}^\bullet$, or an $\mathbf{F}^\bullet$-*essential subvariety*, provided that there is a irreducible component C of $|\operatorname{gecc}^\bullet(\mathbf{F}^\bullet)|$ such that $Y = \eta(C)$.

A connected submanifold $M \subseteq \mathcal{U}$ is an *essential submanifold for* $\mathbf{F}^\bullet$, or an $\mathbf{F}^\bullet$-*essential submanifold*, provided that there is an $\mathbf{F}^\bullet$-essential subvariety Y such that $M = Y_{\mathrm{reg}}$.

A stratum S_α is $\mathbf{F}^\bullet$-*visible* provided that $\overline{S_\alpha}$ is an $\mathbf{F}^\bullet$-essential subvariety. This is equivalent to saying that S_α has a non-zero Morse module, with respect to $\mathbf{F}^\bullet$, in some degree, i.e., provided that $\overline{T^*_{S_\alpha}\mathcal{U}}$ is a component of $\mathrm{gecc}^\bullet(\mathbf{F}^\bullet)$ or, equivalently, that $\overline{T^*_{S_\alpha}\mathcal{U}}$ is an irreducible component of $|\,\mathrm{gecc}^\bullet(\mathbf{F}^\bullet)|$.

Remark 2.8. In terms of the definitions in 2.7, the last equality of Proposition 2.5 can be restated as

$$\mathrm{supp}(\mathbf{F}^\bullet) \ = \ \bigcup_{\substack{\mathbf{F}^\bullet-\text{essential}\\ \text{subvarieties } Y}} Y \ = \ \bigcup_{\substack{\mathbf{F}^\bullet-\text{essential}\\ \text{submanifolds } M}} M \ = \ \bigcup_{\mathbf{F}^\bullet-\text{visible } S_\alpha} \overline{S_\alpha}.$$

§3. Enriched Forms of Previous Results

In this section, we describe the enriched versions of a number of known results. We provide no proofs, since all one has to do is rewrite previous Morse-theoretic proofs, using enriched cycles in place of ordinary cycles.

We continue with all of our notation from the previous two sections.

Definition 3.1. If M is an analytic submanifold of $\mathcal{U}$ and $M \subseteq X$, then the *relative conormal space (of M with respect to f in $\mathcal{U}$)*, $T^*_{f_{|_M}}\mathcal{U}$, is given by

$$T^*_{f_{|_M}}\mathcal{U} := \{(\mathbf{x}, \omega) \in T^*\mathcal{U} \mid \mathbf{x} \in M,\ \omega\big(\ker d_{\mathbf{x}}(f_{|_M})\big) = 0\} =$$

$$\{(\mathbf{x}, \omega) \in T^*\mathcal{U} \mid \mathbf{x} \in M,\ \omega\big(T_{\mathbf{x}}M \cap \ker d_{\mathbf{x}}\tilde{f}\big) = 0\}.$$

If $\mathrm{gecc}^k(\mathbf{F}^\bullet) = \sum E^k_\alpha\left[\overline{T^*_{S_\alpha}\mathcal{U}}\right]$, then we define the *relative graded enriched conormal cycle,* $\big(T^*_{f,\mathbf{F}^\bullet}\mathcal{U}\big)^\bullet$*, of f with respect to* $\mathbf{F}^\bullet$ by

$$\big(T^*_{f,\mathbf{F}^\bullet}\mathcal{U}\big)^k := \sum_{f_{|_{S_\alpha}} \neq \text{constant}} E^k_\alpha\left[\overline{T^*_{f_{|_{S_\alpha}}}\mathcal{U}}\right].$$

The following result is an enriched version of Theorem 2.3 of [**M1**]; it follows trivially by "enriching" the cycles in the Morse theoretic proof given in [**M1**]. Alternatively, one can use the device of perverse cohomology, as in [**M9**], to obtain the result directly from the statement of Theorem 2.3 of [**M1**].

Theorem 3.2. *There is an equality of graded enriched cycles given by*

$$\mathrm{gecc}^k(\psi_f[-1]\mathbf{F}^\bullet) \ = \ \big(T^*_{f,\mathbf{F}^\bullet}\mathcal{U}\big)^k \odot (V(f) \times \mathbb{C}^{n+1}).$$

In particular, $\mathrm{supp}(\psi_f[-1]\mathbf{F}^\bullet) \ = \ \Bigg(V(f) \cap \bigcup_{\substack{\mathbf{F}^\bullet-\text{essential}\\ \text{subvarieties } Y\\ Y \not\subseteq V(f)}} Y\Bigg) \ = \ V(f) \cap \big(\overline{\mathrm{supp}(\mathbf{F}^\bullet) - V(f)}\big).$

The next theorem is the main result of [**M6**], stated in a form that uses our current terminology. One could also obtain this result by using enriched cycles throughout the proof in [**L**].

Theorem 3.3. *Suppose that* $f(\mathbf{p}) = v$.

Then, $\dim_{\mathbf{p}}(\operatorname{supp}\phi_{f-v}[-1]\mathbf{F}^\bullet) \leqslant 0$ *if and only if* $\dim_{\mathbf{p}} \eta\big(|\operatorname{gecc}^\bullet(\mathbf{F}^\bullet)| \cap \operatorname{im} d\tilde{f}\big) \leqslant 0$*, and when this is the case,* $\dim_{(\mathbf{p}, d_{\mathbf{p}}\tilde{f})}\big(|\operatorname{gecc}^\bullet(\mathbf{F}^\bullet)| \cap \operatorname{im} d\tilde{f}\big) \leqslant 0$ *and*

$$H^k(\phi_{f-v}[-1]\mathbf{F}^\bullet)_{\mathbf{p}} \cong \big(\operatorname{gecc}^k(\mathbf{F}^\bullet) \ \odot \ \operatorname{im} d\tilde{f}\big)_{(\mathbf{p}, d_{\mathbf{p}}\tilde{f})},$$

where $\operatorname{im} d\tilde{f}$ *is considered as a graded, enriched cycle.*

The next theorem follows from Proposition 2.2 and Theorem 3.1 of [**M9**], but we could have obtained the result directly by using enriched cycles throughout the proof of Theorem 2.10 of [**M1**].

Theorem 3.4. *There is an equality of closed subsets of* X *given by*

$$\bigcup_{v\in\mathbb{C}} \operatorname{supp}\phi_{f-v}[-1]\mathbf{F}^\bullet \ = \ \eta\big(|\operatorname{gecc}^\bullet(\mathbf{F}^\bullet)| \cap \operatorname{im} d\tilde{f}\big),$$

and, for all k*, an equality of graded, enriched cycles given by*

$$\sum_{v\in\mathbb{C}} \mathbb{P}\big(\operatorname{gecc}^k(\phi_{f-v}[-1]\mathbf{F}^\bullet)\big) \ = \ \pi_*\big(\operatorname{Ex}_{\operatorname{im} d\tilde{f}}(\operatorname{gecc}^k(\mathbf{F}^\bullet))\big).$$

In particular, for all k*, there is an equality of sets*

$$\bigcup_{v\in\mathbb{C}} \eta\big(|\operatorname{gecc}^k(\phi_{f-v}[-1]\mathbf{F}^\bullet)|\big) \ = \ \eta\big(|\operatorname{gecc}^k(\mathbf{F}^\bullet)| \cap \operatorname{im} d\tilde{f}\big).$$

Theorem 3.3 can be recovered very quickly as a special case of Theorem 3.4.

The following corollary is immediate from Theorem 3.4.

Corollary 3.5. *Suppose that* $f(\mathbf{p}) = 0$*, and that* M *is an* R*-module.*

If the enriched cycle $M\big[\{\mathbf{p}\}\times\mathbb{C}^{n+1}\big]$ *is a summand of* $\operatorname{gecc}^k(\mathbf{F}^\bullet)$*, then* $M\big[\{\mathbf{p}\}\times\mathbb{C}^{n+1}\big]$ *is a summand of* $\operatorname{gecc}^k(\phi_f[-1]\mathbf{F}^\bullet)$*. In particular, if* $\{\mathbf{p}\}\times\mathbb{C}^{n+1}$ *is not a component of* $\big|\operatorname{gecc}^\bullet(\phi_f[-1]\mathbf{F}^\bullet)\big|$*, then it is not a component of* $\big|\operatorname{gecc}^\bullet(\mathbf{F}^\bullet)\big|$*.*

Corollary 3.6. *Suppose that* $\phi_f[-1]\mathbf{F}^\bullet = 0$*.*

Then, $\psi_f[-1]\mathbf{F}^\bullet \cong \mathbf{F}^\bullet_{|_{V(f)}}[-1]$ *and*

$$\operatorname{supp}(\psi_f[-1]\mathbf{F}^\bullet) = \operatorname{supp}(\mathbf{F}^\bullet_{|_{V(f)}}[-1]) = V(f) \cap \operatorname{supp}\mathbf{F}^\bullet;$$

in addition, $\dim\big(V(f) \cap \operatorname{supp}\mathbf{F}^\bullet\big) \leqslant -1 + \dim(\operatorname{supp}\mathbf{F}^\bullet)$*. (In particular, if* $\dim(\operatorname{supp}\mathbf{F}^\bullet) = 0$*, then* $V(f) \cap \operatorname{supp}\mathbf{F}^\bullet = \emptyset$*.)*

Proof. The isomorphism is immediate from the fundamental distinguished triangle relating the nearby and vanishing cycles. By 3.4, since $\phi_f[-1]\mathbf{F}^\bullet = 0$, $V(f)$ can not contain an $\mathbf{F}^\bullet$-visible stratum. Therefore, by the final statement of 3.2, $\mathrm{supp}(\psi_f[-1]\mathbf{F}^\bullet) = V(f) \cap \mathrm{supp}\,\mathbf{F}^\bullet$, and the dimension of this must drop. □

Corollary 3.7. *Suppose that f is in the square of the maximal ideal of X at $\mathbf{p}$, i.e., $f \in \mathfrak{m}^2_{X,\mathbf{p}}$. If $\phi_f[-1]\mathbf{F}^\bullet$ is perverse in a neighborhood of $\mathbf{p}$, then $\mathbf{F}^\bullet$ is perverse in a neighborhood of $\mathbf{p}$. If $\mathbf{p}$ is not in the support of $\phi_f[-1]\mathbf{F}^\bullet$, then $\mathbf{p}$ is not in the support of $\mathbf{F}^\bullet$.*

Proof. Since $f \in \mathfrak{m}^2_{X,\mathbf{p}}$, it follows that, for all S_α, if $\mathbf{p} \in \overline{S_\alpha}$, then $(\mathbf{p}, d_{\mathbf{p}}\tilde{f}) \in \overline{T^*_{S_\alpha}\mathcal{U}}$. Therefore, Theorem 3.4 implies that, if $\mathrm{gecc}^\bullet(\phi_f[-1]\mathbf{F}^\bullet)$ is concentrated in degree 0, then $\mathrm{gecc}^\bullet(\mathbf{F}^\bullet)$ must be concentrated in degree 0 over a neighborhood of $\mathbf{p}$. This yields the first statement. The second statement uses the same argument. □

As our final result of this section, we need to state the enriched version of Theorem I.2.20 of [**M7**]. We will use this result in Section 6, where it will enable us to actually perform calculations. We state the result in the form in which we shall use it. Recall the definition of the graded, enriched exceptional divisor from 2.2.

Theorem 3.8. *Let $C^\bullet$ be a purely $(n+1)$-dimensional graded, enriched cycle on $\mathcal{U} \times \mathbb{C}^{n+1}$. Let $\mathbf{h} := (h_0, \dots, h_n)$ be an $(n+1)$-tuple of analytic functions on $\mathcal{U} \times \mathbb{C}^{n+1}$.*

Suppose that, for all j such that $0 \leqslant j \leqslant n+1$, $\mathrm{Ex}^\bullet_{\mathbf{h}}(C^\bullet)$ properly intersects $\mathcal{U} \times \mathbb{C}^{n+1} \times \mathbb{P}^j \times \{\mathbf{0}\}$ inside $\mathcal{U} \times \mathbb{C}^{n+1} \times \mathbb{P}^n$, and define ${}^k\Delta^j_{\mathbf{h}}$ by

$$ {}^k\Delta^j_{\mathbf{h}} = \xi_*\big(\mathrm{Ex}^k_{\mathbf{h}}(C^\bullet) \ \odot\ (\mathcal{U} \times \mathbb{C}^{n+1} \times \mathbb{P}^j \times \{\mathbf{0}\})\big). $$

Then, there exists an open neighborhood of $V(\mathbf{h})$ in $\mathcal{U} \times \mathbb{C}^{n+1}$ in which ${}^k\Delta^j_{\mathbf{h}}$ can be calculated via the following inductive process, in which all of the intersections are proper:

If $C^k = \sum_V C_V[V]$, then let ${}^k\Pi^{n+1}_{\mathbf{h}} := \sum_{V \not\subseteq V(\mathbf{h})} C_V[V]$. Then, one can write

$$ {}^k\Pi^{n+1}_{\mathbf{h}} \odot V(h_n) = {}^k\Pi^n_{\mathbf{h}} + {}^k\Delta^n_{\mathbf{h}}, $$

where ${}^k\Pi^n_{\mathbf{h}}$ denotes the sum of all those components (with their component modules) of the intersection which are not contained (as sets) in $V(\mathbf{h})$, and ${}^k\Delta^n_{\mathbf{h}}$ will precisely equal the sum of the components of the intersection which are contained in $V(\mathbf{h})$. Proceeding inductively, one writes the intersection

$$ {}^k\Pi^{j+1}_{\mathbf{h}} \odot V(h_j) = {}^k\Pi^j_{\mathbf{h}} + {}^k\Delta^j_{\mathbf{h}}, $$

where ${}^k\Pi^j_{\mathbf{h}}$ denotes the sum of all those components of the intersection which are not contained in $V(\mathbf{h})$, and ${}^k\Delta^i_{\mathbf{h}}$ will precisely equal the sum of the components of the intersection which are contained in $V(\mathbf{h})$.

§4. **Partitions and Stratifications**

In defining $\mathrm{gecc}^\bullet(\mathbf{F}^\bullet)$, we used a Whitney stratification of X with respect to which $\mathbf{F}^\bullet$ was constructible. However, in practice, we do not explicitly need Whitney's condition b) or the condition of the frontier. Since weaker hypotheses on the "stratifications" will yield stronger, more useful results, we wish to describe the types of partitions of X that we will use, and prove a few basic results.

We continue with all of the notation from the previous sections **except that** $\{S_\alpha\}$ **is no longer assumed to be a Whitney stratification**. Throughout the remainder of this paper, if Y is an analytic set, then when we write $\dim_{\mathbf{p}} Y \leqslant 0$, we mean that either $\mathbf{p}$ is an isolated point of Y ($\dim_{\mathbf{p}} Y = 0$) or $\mathbf{p} \notin Y$ ($\dim_{\mathbf{p}} Y = -\infty$).

Definition 4.1. A *(complex analytic) partition of* X is a locally-finite collection $\mathcal{S} := \{S_\alpha\}_\alpha$ of disjoint, connected, complex analytic manifolds whose union is all of X and such that, for all α, $\overline{S_\alpha}$ and $\overline{S_\alpha} - S_\alpha$ are complex analytic subsets of $\mathcal{U}$. We still refer to the elements of $\mathcal{S}$ as *strata*.

The strata of a partition are partially-ordered by: $S_\beta \leqslant S_\alpha$ if and only if $S_\beta \subseteq \overline{S_\alpha}$.

A partition $\mathcal{S}$ of X is a *stratification* provided that the condition of the frontier holds, i.e., for all $S_\alpha \in \mathcal{S}$, $\overline{S_\alpha}$ is a union of strata. This means that $S_\beta \leqslant S_\alpha$ if and only if $S_\beta \cap \overline{S_\alpha} \neq \emptyset$.

A partition $\mathcal{S}$ of X is a *Whitney a) partition* provided that all pairs of strata satisfy Whitney's condition a); in conormal terms this means that, for all $S_\beta \in \mathcal{S}$,

$$\overline{T^*_{S_\beta}\mathcal{U}} \ \subseteq \ \bigcup_\alpha T^*_{S_\alpha}\mathcal{U}.$$

In particular, this implies that $\bigcup_\alpha T^*_{S_\alpha}\mathcal{U}$ is equal to $\bigcup_\alpha \overline{T^*_{S_\alpha}\mathcal{U}}$ and, hence, is closed.

If $\mathbf{F}^\bullet$ is a bounded, constructible complex of sheaves on X, then a partition $\mathcal{S}$ of X is an $\mathbf{F}^\bullet$*-partition of* X provided that

$$|\operatorname{gecc}^\bullet(\mathbf{F}^\bullet)| \ \subseteq \ \bigcup_\alpha \overline{T^*_{S_\alpha}\mathcal{U}}.$$

In an $\mathbf{F}^\bullet$-partition of X, a *visible stratum*, S_α, is one such that $\overline{T^*_{S_\alpha}\mathcal{U}} \subseteq |\operatorname{gecc}^\bullet(\mathbf{F}^\bullet)|$. Note that this extends our definition in 2.7 to the case where $\mathcal{S}$ need not be a Whitney stratification. Also note that, if $\mathcal{S}$ is an $\mathbf{F}^\bullet$-partition of X, then, as before, Proposition 2.5 implies that

$$\operatorname{supp}(\mathbf{F}^\bullet) = \bigcup_{\mathbf{F}^\bullet-\text{visible } S_\alpha} \overline{S_\alpha};$$

in particular, this implies that the maximal elements of $\{S_\alpha \in \mathcal{S} \mid S_\alpha \subseteq \operatorname{supp}(\mathbf{F}^\bullet)\}$ are all visible, and that their union is equal to $\operatorname{supp}(\mathbf{F}^\bullet)$.

Certainly, a Whitney stratification of X, with connected strata, with respect to which $\mathbf{F}^\bullet$ is constructible, is an $\mathbf{F}^\bullet$-partition of X. However, in our results, it is only the properties of $\mathbf{F}^\bullet$-partitions that we actually use.

Definition 4.2. Suppose that $\mathcal{S}$ is a partition of X. Let $S_\alpha \in \mathcal{S}$, and let $\mathbf{p} \in X$ (but $\mathbf{p}$ is not necessarily in S_α).

The germ at $\mathbf{p}$ of an analytic submanifold $N \subseteq \mathcal{U}$ is *essentially transverse to* S_α *at* $\mathbf{p}$ (in $\mathcal{U}$) provided that there is an open neighborhood of $\mathbf{p}$ in which N transversely intersects $S_\alpha - \{\mathbf{p}\}$ inside $\mathcal{U}$.

We say that the germ N is *essentially transverse to* $\mathcal{S}$ *at* $\mathbf{p}$ provided that N is essentially transverse to S_α at $\mathbf{p}$, for all $S_\alpha \in \mathcal{S}$.

If N is essentially transverse to $\mathcal{S}$ at $\mathbf{p}$, then, in a neighborhood of $\mathbf{p}$, there is an *induced partition of* $X \cap N$ in which the strata are the point-stratum $\{\mathbf{p}\}$, together with the connected-components of the intersection $(S_\alpha - \{\mathbf{p}\}) \cap N$.

The coordinates $\mathbf{z}$ for $\mathcal{U}$ are *essentially transverse to* S_α *at* $\mathbf{p}$ provided that, for all i with $0 \leqslant i \leqslant n$, $V(z_0 - p_0, \dots, z_i - p_i)$ is essentially transverse to S_α at $\mathbf{p}$. This is clearly equivalent to requiring that, for all i with $0 \leqslant i \leqslant -1 + \dim S_\alpha$, $V(z_0 - p_0, \dots, z_i - p_i)$ is essentially transverse to S_α at $\mathbf{p}$.

The coordinates $\mathbf{z}$ for $\mathcal{U}$ are *essentially transverse to* $\mathcal{S}$ *at* $\mathbf{p}$ provided that $\mathbf{z}$ is essentially transverse to each S_α at $\mathbf{p}$. This is equivalent to: $V(z_0 - p_0)$ is essentially transverse to $\mathcal{S}$ at $\mathbf{p}$ and, for all i such that $1 \leqslant i \leqslant n$, $V(z_i - p_i)$ is essentially transverse to the induced partition of $X \cap V(z_0 - p_0, \dots, z_{i-1} - p_{i-1})$.

In the proposition below, we should point out that the only difference between the statements in b) and c) is that, in b), $V(z_i - p_i)$ explicitly appears in the intersection, while it does not in c).

Proposition 4.3. *Suppose that* $\mathcal{S}$ *is a partition of* X. *Let* $S_\alpha \in \mathcal{S}$, *and let* $\mathbf{p} \in X$. *Then, the following are equivalent:*

a) The coordinates $\mathbf{z}$ *are essentially transverse to* S_α *at* $\mathbf{p}$;

b) there exists an open neighborhood $\mathcal{W}$ *of* $\mathbf{p}$ *in* $\mathcal{U}$ *such that, for all* i *such that* $0 \leqslant i \leqslant n$,

$$\mathbb{P}(T^*_{S_\alpha}\mathcal{U}) \ \cap \ \Big(\big(\mathcal{W} \cap V(z_0 - p_0, \dots, z_i - p_i)\big) \times \mathbb{P}^i \times \{\mathbf{0}\}\Big) \ \subseteq \ \{\mathbf{p}\} \times \mathbb{P}^i \times \{\mathbf{0}\}.$$

c) there exists an open neighborhood $\mathcal{W}$ *of* $\mathbf{p}$ *in* $\mathcal{U}$ *such that, for all* i *such that* $0 \leqslant i \leqslant n$,

$$\mathbb{P}(T^*_{S_\alpha}\mathcal{U}) \ \cap \ \Big(\big(\mathcal{W} \cap V(z_0 - p_0, \dots, z_{i-1} - p_{i-1})\big) \times \mathbb{P}^i \times \{\mathbf{0}\}\Big) \ \subseteq \ \{\mathbf{p}\} \times \mathbb{P}^i \times \{\mathbf{0}\}.$$

(When $i = 0$, *we mean that* $\mathbb{P}(T^*_{S_\alpha}\mathcal{U}) \ \cap \ \big(\mathcal{W} \times \mathbb{P}^0 \times \{\mathbf{0}\}\big) \ \subseteq \ \{\mathbf{p}\} \times \mathbb{P}^0 \times \{\mathbf{0}\}$.*)*

Moreover, we obtain equivalent statements if we replace the condition that $0 \leqslant i \leqslant n$ *in b) and c) by the condition that* $0 \leqslant i \leqslant -1 + \dim S_\alpha$.

Proof. That a) and b) are equivalent is simply a translation of transversality into conormal terms. Certainly, c) implies b). That we obtain equivalent statements if we replace $0 \leqslant i \leqslant n$ in b) and c) by the condition that $0 \leqslant i \leqslant -1 + \dim S_\alpha$ is also trivial. The surprising implication is that b) implies the seemingly stronger c).

Assume that b) holds. Suppose that we had the germ of a complex analytic curve $\mathbf{q}(t) := (\mathbf{x}(t), [\omega(t)]) \in \mathcal{U} \times \mathbb{P}^n$ such that $\mathbf{x}(0) = \mathbf{p}$ and so that, for small $t \neq 0$, $\mathbf{q}(t)$ is in

$$\mathbb{P}(T^*_{S_\alpha}\mathcal{U}) \ \cap \ \big(V(z_0 - p_0, \dots, z_{k-1} - p_{k-1}) \times \mathbb{P}^k \times \{\mathbf{0}\}\big).$$

Then, for $t \neq 0$, $\mathbf{x}(t) = (\mathbf{0}, x_k(t), \dots, x_n(t))$, $\omega(t) = \omega_0(t)dz_0 + \dots \omega_k(t)dz_k$, $\mathbf{x}'(t) \in T_{\mathbf{x}(t)}S_\alpha$, and so $\omega_k(t)x'_k(t) \equiv 0$. Thus, either $\omega_k(t) \equiv 0$ or $x_k(t) \equiv p_k$.

Letting $Y_k := V(z_0 - p_0, \dots, z_k - p_k) \times \mathbb{P}^k \times \{\mathbf{0}\}$, the above paragraph shows that, as germs over $\mathbf{p}$,

$$\mathbb{P}(T^*_{S_\alpha}\mathcal{U}) \ \cap\ \big(V(z_0 - p_0, \dots, z_{k-1} - p_{k-1}) \times \mathbb{P}^k \times \{\mathbf{0}\}\big) \ \subseteq\ Y_{k-1} \ \cup\ Y_k,$$

where $Y_{-1} := \emptyset$. Now, b) yields a contradiction. □

Corollary 4.4. *Suppose that $\mathcal{S}$ is a Whitney a) partition of X. Then the coordinates $\mathbf{z}$ are essentially transverse to $\mathcal{S}$ at $\mathbf{p} \in X$ if and only if, for all $S_\alpha \in \mathcal{S}$, for all i such that $0 \leqslant i \leqslant n$,*

$$\dim_{\mathbf{p}} \Big(V(z_0 - p_0, \dots, z_{i-1} - p_{i-1}) \ \cap\ \nu\big(\mathbb{P}(\overline{T^*_{S_\alpha}\mathcal{U}}) \ \cap\ (\mathcal{U} \times \mathbb{P}^i \times \{\mathbf{0}\})\big)\Big) \ \leqslant\ 0.$$

In particular, if $\mathbf{z}$ is essentially transverse to $\mathcal{S}$ at $\mathbf{p}$, then $\mathbf{z}$ is essentially transverse to $\mathcal{S}$ at all points near $\mathbf{p}$.

Proof. This follows from 4.3.c and the fact that the Whitney a) condition implies that $\bigcup_\alpha T^*_{S_\alpha}\mathcal{U} = \bigcup_\alpha \overline{T^*_{S_\alpha}\mathcal{U}}$. □

As our final result of this section, we wish to characterize Thom's a_f condition in terms of exceptional divisors; this is essentially Proposition 4.3 from [**M9**]. Recall the definition of the relative conormal space from 3.1.

Definition 4.5. Let M and N be analytic submanifolds of X such that f has constant rank on N. Then, the pair (M, N) *satisfies Thom's a_f condition at a point* $\mathbf{x} \in N$ if and only if we have the containment $\Big(\overline{T^*_{f_{|_M}}\mathcal{U}}\Big)_{\mathbf{x}} \ \subseteq\ \Big(T^*_{f_{|_N}}\mathcal{U}\Big)_{\mathbf{x}}$ of fibres over $\mathbf{x}$.

In particular, if f is, in fact, constant on N, then the pair (M, N) satisfies Thom's a_f condition at a point $\mathbf{x} \in N$ if and only if we have the containment $\Big(\overline{T^*_{f_{|_M}}\mathcal{U}}\Big)_{\mathbf{x}} \ \subseteq\ \Big(T^*_N\mathcal{U}\Big)_{\mathbf{x}}$ of fibres over $\mathbf{x}$. If f is constant on both M and N, then the a_f condition reduces to Whitney's condition a).

Remark 4.6. Note that, if Y is an analytic subspace of X, and M is an open dense subset of Y_{reg}, then $\overline{T^*_{Y_{\text{reg}}}\mathcal{U}} = \overline{T^*_M\mathcal{U}}$ and, for every submanifold $N \subseteq \mathcal{U}$ and every $\mathbf{x} \in N$, (Y_{reg}, N) satisfies Whitney's condition a) at $\mathbf{x}$ if and only if (M, N) satisfies Whitney's condition a) at $\mathbf{x}$; moreover, if f is not constant on any irreducible component of Y, then (Y_{reg}, N) satisfies Thom's a_f condition at $\mathbf{x}$ if and only if (M, N) satisfies Thom's a_f condition at $\mathbf{x}$. Thus, below, it will suffice to work with Y_{reg} everywhere, instead of the seemingly more general M.

The notion of Thom's a_f condition that we use below (and above) is slightly more general than is sometimes the case; we do not require the rank of f to be constant on the bigger stratum. If we were to require the rank of f to be constant on the bigger stratum, then we would be forced to write the more cumbersome " $(Y_{\text{reg}} - \Sigma(f_{|_{Y_{\text{reg}}}}), N)$ satisfies Thom's a_f condition at $\mathbf{x}$". Moreover, if C is a component of Y on which f is constant, then saying that (Y_{reg}, N) satisfies the a_f condition implies that (C_{reg}, N) satisfies Whitney's condition a); the condition that $(Y_{\text{reg}} - \Sigma(f_{|_{Y_{\text{reg}}}}), N)$ satisfies Thom's a_f condition would ignore what happens on a component such as C.

Definition 4.7. An *a_f partition of X with respect to* $\mathbf{F}^\bullet$ (or, an *$a_{f,\mathbf{F}^\bullet}$ partition of* X) is an $\mathbf{F}^\bullet$-partition, $\mathcal{S}$, of X such that $V(f)$ is a union is strata (the $V(f)$ *strata*) and such that for every $\mathbf{F}^\bullet$-visible S_α and every $V(f)$ stratum S_β, the pair (S_α, S_β) satisfies the a_f condition.

The following theorem looks like a significant improvement of Proposition 4.3 of [**M9**]; however, the proof is essentially the same. In Section 6, this theorem will enable us to link Thom's a_f condition with the vanishing cycles along f.

Theorem 4.8. *Suppose that Y is an analytic subset of X. Let E denote the exceptional divisor in $\mathrm{Bl}_{\mathrm{im}\, d\tilde f}\, \overline{T^*_{Y_{\mathrm{reg}}}\mathcal{U}} \subseteq \mathcal{U} \times \mathbb{C}^{n+1} \times \mathbb{P}^n$. Suppose that $N \subseteq X$ is a complex analytic submanifold of $\mathcal{U}$ and that $\mathbf{x} \in N$.*

Then, (Y_{reg}, N) satisfies Thom's a_f condition at $\mathbf{x}$ if and only if

i) (Y_{reg}, N) *satisfies Whitney's condition a) at* $\mathbf{x}$*;*

ii) $d_{\mathbf{x}}\tilde f \in \left(T^*_N\mathcal{U}\right)_{\mathbf{x}}$*;*

iii) there is the containment of fibres above $\mathbf{x}$ *given by* $\left(\pi(E)\right)_{\mathbf{x}} \subseteq \left(\mathbb{P}(T^*_N\mathcal{U})\right)_{\mathbf{x}}$.

Proof. We first make a few simple observations.

- Suppose (Y_{reg}, N) satisfies Thom's a_f condition at $\mathbf{x}$.

Then certainly i) and ii) follow; for we can use as relative conormal covectors every conormal to Y_{reg} and every covector of the form $d_{\mathbf{y}}\tilde f$, where $\mathbf{y} \in Y_{\mathrm{reg}}$. In addition, we proved in Proposition 4.3 of [**M9**] that the a_f condition implies that

$$\left(\pi(E)\right)_{\mathbf{x}} \subseteq \left(\mathbb{P}(T^*_N\mathcal{U})\right)_{\mathbf{x}}.$$

This proves one direction of the theorem.

- Now suppose that i), ii), and iii) hold.

If C is a component of Y on which f is constant, then limiting relative conormals to C_{reg} are the same as limiting (absolute) conormals to C_{reg}. As we are assuming that Whitney's condition a) holds, it follows that (C_{reg}, N) satisfies that a_f at $\mathbf{x}$.

For components C of Y on which f is non-constant, the proof of Proposition 4.3 of [**M9**] shows that (C_{reg}, N) satisfies that a_f at $\mathbf{x}$. □

§5. **Characteristic Polar Complexes, Modules, and Isolating Coordinates**

In this section, we will define one our primary objects of study – *the characteristic polar modules* – and relate them to polar varieties, graded, enriched characteristic cycles, and iterated vanishing and nearby cycles. This will require us to investigate how generic a linear choice of coordinates must be in order to produce nice results. It is important throughout this section, and throughout the remainder of this paper, that our notion of *isolating coordinates* is an **effective** notion of "generic"; that is, in many situations, one can determine fairly easily whether the coordinates are isolating. Moreover, if the coordinates are isolating at a given point, then they are isolating at all nearby points – there is no need to re-choose the coordinates at each point. These properties are important, since our goal is to produce effectively calculable data that one can associate to a singularity.

We continue with all of our previous notation, and also introduce new notation.

We let Y be a new complex analytic subspace of $\mathcal{U}$ and we let $\mathbf{A}^\bullet$ be a bounded, constructible complex of R-modules on Y. Let $\mathcal{R} := \{R_\beta\}$ be an $\mathbf{A}^\bullet$-partition of Y. Our reason for introducing these new objects is that, later, we will return to the setting of the previous sections by considering the special case where $Y = V(f) = X \cap V(\tilde{f})$ and $\mathbf{A}^\bullet = \phi_f[-1]\mathbf{F}^\bullet$.

Fix a point $\mathbf{p} = (p_0, \dots, p_n) \in Y$, and let $d := \dim_{\mathbf{p}}(\operatorname{supp} \mathbf{A}^\bullet)$.

Recall that we use $\mathbf{z} = (z_0, \dots, z_n)$ to denote coordinates on $\mathcal{U}$. Below, when the context makes the domains clear, we shall not distinguish in the notation between the coordinate functions z_i and their restrictions to various subspaces. As we will be projectivizing conormal varieties, we assume that $\operatorname{codim}_{\mathcal{U}} Y \geqslant 1$.

Throughout the remainder of this paper, it will be convenient to adopt the standard convention that, when $m = 0$, $V(z_0 - p_0, \dots, z_{m-1} - p_{m-1}) = \mathcal{U}$.

This section contains a number of technical results. However, the reader should note that the main points of this section are:

- In Definition 5.1, we define the *j-th characteristic polar module in degree k of $\mathbf{A}^\bullet$ with respect to the coordinates $\mathbf{z}$ at the point $\mathbf{p}$* to be

$$^k\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p}) \ := \ H^k\big(\phi_{z_j - p_j}[-1]\psi_{z_{j-1}-p_{j-1}}[-1]\dots\psi_{z_0-p_0}[-1]\mathbf{A}^\bullet\big)_{\mathbf{p}}.$$

- The coordinates $\mathbf{z}$ are *$\mathbf{A}^\bullet$-isolating at $\mathbf{p}$* provided that the support of

$$\phi_{z_j - p_j}[-1]\psi_{z_{j-1}-p_{j-1}}[-1]\dots\psi_{z_0-p_0}[-1]\mathbf{A}^\bullet$$

is at most 0-dimensional at $\mathbf{p}$ for all j. Despite the fact that this condition looks fairly unmanageable, in Theorem 5.10, we prove that this condition has a nice interpretation in terms of intersections with $\operatorname{gecc}^\bullet(\mathbf{A}^\bullet)$. Moreover, under the assumption that the coordinates are $\mathbf{A}^\bullet$-isolating, general results on nearby cycles, vanishing cycles, and perverse cohomology allow us to conclude Theorem 5.18: a result which yields chain complexes, containing the $^k\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{x})$, whose cohomology is isomorphic to the stalk cohomology of the perverse cohomology of $\mathbf{A}^\bullet$ in each degree. This result is important because the modules appearing in these chain complexes can be calculated; see the following paragraph.

- In Definition 5.20, under the assumption that the coordinates are $\mathbf{A}^\bullet$-isolating, we define the *enriched j-th characteristic polar cycle in degree k of $\mathbf{A}^\bullet$ with respect to the coordinates* $\mathbf{z}$ to be

$$^k\Gamma^j_{\mathbf{A}^\bullet,\mathbf{z}} := \nu_*\Big(\mathbb{P}(\operatorname{gecc}^k(\mathbf{A}^\bullet)) \ \odot \ \mathcal{W} \times \mathbb{P}^j \times \{\mathbf{0}\}\Big)$$

and we show in Theorem 5.23 that $^k\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{x})$ can, in fact, be calculated by

$$^k\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p}) \ \cong \ \big(^k\Gamma^j_{\mathbf{A}^\bullet,\mathbf{z}} \ \odot \ V(z_0 - p_0, \dots, z_{j-1} - p_{j-1})\big)_{\mathbf{p}}.$$

Definition 5.1. For all j such that $0 \leqslant j \leqslant n$, for all $\mathbf{a} \in \mathbb{C}^{j+1}$, we define the *$j$-th characteristic polar complex of $\mathbf{A}^\bullet$ with respect to $\mathbf{z}$ at $\mathbf{a}$* to be

$$\Phi^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{a}) \ := \ \phi_{z_j - a_j}[-1]\psi_{z_{j-1}-a_{j-1}}[-1]\dots\psi_{z_0-a_0}[-1]\mathbf{A}^\bullet.$$

When $j=0$, we mean that $\Phi^0_{\mathbf{A}^\bullet,\mathbf{z}}(a_0) \ := \ \phi_{z_0-a_0}[-1]\mathbf{A}^\bullet$. For $\mathbf{x}\in\mathbb{C}^{n+1}$, it is convenient to define $\Phi^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{x}) \ := \ \Phi^j_{\mathbf{A}^\bullet,\mathbf{z}}(x_0,x_1,\dots,x_j)$.

For all $\mathbf{x}\in Y$, we define the *j-th characteristic polar module in degree k of $\mathbf{A}^\bullet$ with respect to $\mathbf{z}$ at $\mathbf{x}$*, ${}^k\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{x})$, to be the degree k stalk cohomology at $\mathbf{x}$ of $\Phi^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{x})$, i.e.,

$$ {}^k\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{x}) \ := \ H^k\big(\Phi^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{x})\big)_{\mathbf{x}}. $$

The *support of* ${}^k\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}$ is, naturally, defined to be the closure of the set $\{\mathbf{x}\in Y \mid {}^k\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{x})\neq 0\}$. We denote this support by $\operatorname{supp}({}^k\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}})$.

We define the *support of* ${}^\bullet\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}$ to be the closure of the set

$$ \{\mathbf{x}\in Y \mid \text{there exists } k \text{ such that } \ {}^k\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{x})\neq 0\}. $$

We denote this support by $\operatorname{supp}({}^\bullet\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}})$. By boundedness, there are a finite number of k such that ${}^k\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}\neq 0$; hence,

$$ \operatorname{supp}({}^\bullet\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}) \ = \ \bigcup_k \operatorname{supp}({}^k\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}). $$

If the base ring R is a domain, we define the *j-th polar Euler number of $\mathbf{A}^\bullet$ with respect to $\mathbf{z}$ at $\mathbf{x}$* to be

$$ {}^{\mathrm{ord}}\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{x}) = \sum_k (-1)^k \operatorname{rk}\big({}^k\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{x})\big). $$

We make the following trivial observation.

Proposition 5.2. *Let $\tilde{\mathbf{z}}$ denote the "rotated" coordinate system $(z_1,z_2,\dots,z_n,z_0)$. Fix x_0. For all j such that $1\leqslant j\leqslant n$, for all $\mathbf{x}\in V(z_0-x_0)$,*

$$ \Phi^{j-1}_{\psi_{z_0-x_0}[-1]\mathbf{A}^\bullet,\tilde{\mathbf{z}}}(\mathbf{x}) = \Phi^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{x}), $$

and, for all k,

$$ {}^k\gamma^{j-1}_{\psi_{z_0-x_0}[-1]\mathbf{A}^\bullet,\tilde{\mathbf{z}}}(\mathbf{x}) \ = \ {}^k\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{x}). $$

Proof. This is immediate from the definitions. □

Only slightly less trivial is the following.

Proposition 5.3. *If R is a domain, then, for all m such that $0\leqslant m\leqslant n$, the Euler characteristic of the stalk of $\mathbf{A}^\bullet$ at $\mathbf{x}$ is given by*

$$ \chi(\mathbf{A}^\bullet)_{\mathbf{x}} = (-1)^{m+1}\chi(\psi_{z_m-x_m}[-1]\dots\psi_{z_0-x_0}[-1]\mathbf{A}^\bullet)_{\mathbf{x}} \ + \ \sum_{j\leqslant m}(-1)^j\big({}^{\mathrm{ord}}\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{x})\big). $$

Proof. This follows immediately by inductively applying the fact that Euler characteristics are additive over long exact sequences to the long exact sequences on stalk cohomology, at $\mathbf{x}$, which come from the distinguished triangles

$$\mathbf{A}^\bullet_{|_{V(z_j-x_j)}}[-1] \ \rightarrow\ \psi_{z_j-x_j}[-1]\mathbf{A}^\bullet \ \rightarrow\ \phi_{z_j-x_j}[-1]\mathbf{A}^\bullet \ \xrightarrow{[1]} \mathbf{A}^\bullet_{|_{V(z_j-x_j)}}[-1]. \quad \square$$

While it is clear that

$$\operatorname{supp}(\Phi^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{x})) \subseteq V(z_0-x_0,\dots,z_j-x_j)\cap\operatorname{supp}({}^\bullet\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}),$$

the reverse inclusion need not hold. In order to determine a more precise relationship between $\operatorname{supp}(\Phi^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{x}))$ and $V(z_0-x_0,\dots,z_j-x_j)\cap\operatorname{supp}({}^\bullet\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}})$, we need a lemma and a proposition.

Lemma 5.4. *Let $h: Y\to\mathbb{C}$ be an analytic function, and let $\tilde{h}:\mathcal{U}\to\mathbb{C}$ be a local extension of h near $\mathbf{p}$. Suppose that $\mathbf{p}\notin\operatorname{supp}\phi_{h-h(\mathbf{p})}[-1]\mathbf{A}^\bullet$. Then, for all k, there is an equality of fibres over $\mathbf{p}$ given by*

$$\big|\big(T^*_{h,\mathbf{A}^\bullet}\mathcal{U}\big)^k\big|_{\mathbf{p}} \ = \ |\operatorname{gecc}^k(\mathbf{A}^\bullet)|_{\mathbf{p}} + \ < d_{\mathbf{p}}\tilde{h} >,$$

where $< d_{\mathbf{p}}\tilde{h} >$ denotes the linear subspace of all scalar multiples of $d_{\mathbf{p}}\tilde{h}$.

Proof. By the definition of $T^*_{h_{|_{R_\beta}}}\mathcal{U}$, if $\mathbf{p}\in R_\beta$ and $d_{\mathbf{p}}\tilde{h}\notin T^*_{R_\beta}\mathcal{U}$, then

$$\big(T^*_{h_{|_{R_\beta}}}\mathcal{U}\big)_{\mathbf{p}} = \big(T^*_{R_\beta}\mathcal{U}\big)_{\mathbf{p}} + \langle d_{\mathbf{p}}\tilde{h}\rangle.$$

We need to deal with possible strata R_β such that $\mathbf{p}\in\overline{R_\beta}-R_\beta$.

Fix k. Since $\mathbf{p}\notin\operatorname{supp}\phi_{h-h(\mathbf{p})}[-1]\mathbf{A}^\bullet$, Theorem 3.4 implies that $d_{\mathbf{p}}\tilde{h}\notin|\operatorname{gecc}^k(\mathbf{A}^\bullet)|_{\mathbf{p}}$. Suppose that $\mathbf{p}\in\overline{R_\beta}$ and $\overline{T^*_{R_\beta}\mathcal{U}}$ is a component of $|\operatorname{gecc}^k(\mathbf{A}^\bullet)|$; then, $d_{\mathbf{p}}\tilde{h}\notin\big(\overline{T^*_{R_\beta}\mathcal{U}}\big)_{\mathbf{p}}$.

By 3.4, the set $\eta\big(|\operatorname{gecc}^\bullet(\mathbf{A}^\bullet)|\cap\operatorname{im}d\tilde{h}\big)$ is closed, and so, for all $\mathbf{x}$ near $\mathbf{p}$, $d_{\mathbf{x}}\tilde{h}\notin|\operatorname{gecc}^k(\mathbf{A}^\bullet)|_{\mathbf{x}}$. Therefore, if , we may take $\mathbf{x}_i\in R_\beta$ such that $\mathbf{x}_i\to\mathbf{p}$ and

$$\big(T^*_{h_{|_{R_\beta}}}\mathcal{U}\big)_{\mathbf{x}_i} = \big(T^*_{R_\beta}\mathcal{U}\big)_{\mathbf{x}_i} + \langle d_{\mathbf{x}_i}\tilde{h}\rangle.$$

As $d_{\mathbf{p}}\tilde{h}\notin\big(\overline{T^*_{R_\beta}\mathcal{U}}\big)_{\mathbf{p}}$, we conclude that the limits behave "nicely", and thus

$$\big(\overline{T^*_{h_{|_{R_\beta}}}\mathcal{U}}\big)_{\mathbf{p}} = \big(\overline{T^*_{R_\beta}\mathcal{U}}\big)_{\mathbf{p}} + \langle d_{\mathbf{p}}\tilde{h}\rangle. \quad \square$$

Definition 5.5 For all m such that $0\leqslant m\leqslant n$, we define $\Theta^m_{\mathbf{A}^\bullet,\mathbf{z}}$ by

$$\Theta^m_{\mathbf{A}^\bullet,\mathbf{z}} \ := \ \nu\Big(|\mathbb{P}(\operatorname{gecc}^\bullet(\mathbf{A}^\bullet))| \ \cap \ (\mathcal{U}\times\mathbb{P}^m\times\{\mathbf{0}\})\Big),$$

and we define $\Gamma^m_{\mathbf{A}^\bullet,\mathbf{z}}$ to be the union of the m-dimensional components of $\Theta^m_{\mathbf{A}^\bullet,\mathbf{z}}$. We refer to $\Gamma^m_{\mathbf{A}^\bullet,\mathbf{z}}$ as the *m-dimensional characteristic polar variety of $\mathbf{A}^\bullet$ with respect to $\mathbf{z}$*.

Remark 5.6. Note that each $\Theta^m_{\mathbf{A}^\bullet,\mathbf{z}}$ is closed. In addition, if R_β is a maximal stratum of $\operatorname{supp}\mathbf{A}^\bullet$, and $d_\beta := \dim R_\beta$, then the fibres of $|\mathbb{P}(\operatorname{gecc}^\bullet(\mathbf{A}^\bullet))|$ over R_β will be of dimension $n-d_\beta$, and so these fibres must intersect $\mathbb{P}^{d_\beta}\times\{\mathbf{0}\}$; therefore, $\overline{R_\beta}\subseteq \Theta^{d_\beta}_{\mathbf{A}^\bullet,\mathbf{z}}$.

It follows that, if $d := \dim_{\mathbf{p}}\operatorname{supp}(\mathbf{A}^\bullet)$, then, for all $m \geqslant d$, as germs of sets at $\mathbf{p}$, $\Theta^m_{\mathbf{A}^\bullet,\mathbf{z}} = \operatorname{supp}(\mathbf{A}^\bullet)$.

The sets $\Theta^m_{\mathbf{A}^\bullet,\mathbf{z}}$ are very closely related to the absolute polar varieties of Lê and Teissier. The set $\nu\big(\overline{T^*_{R_\beta}\mathcal{U}} \,\cap\, (\mathcal{U}\times\mathbb{P}^m\times\{\mathbf{0}\})\big)$ consists of the closure of the critical locus of the map $(z_0,\dots,z_m)_{|R_\beta}$ together with some possible "degenerate" critical points on smaller strata. By Whitney's condition a), these degenerate points will be critical points of $(z_0,\dots,z_m)$ restricted to smaller strata. Hence, we have the containments

$$\bigcup_{\substack{\mathbf{A}^\bullet-\text{visible}\\ R_\beta}} \overline{\operatorname{crit}(z_0,\dots,z_m)_{|R_\beta}} \;\subseteq\; \Theta^m_{\mathbf{A}^\bullet,\mathbf{z}} \;\subseteq\; \bigcup_{\substack{\mathbf{A}^\bullet-\text{visible}\\ R_\beta}}\;\bigcup_{R_\gamma\subseteq\overline{R_\beta}} \overline{\operatorname{crit}(z_0,\dots,z_m)_{|R_\gamma}}.$$

If the coordinates $\mathbf{z}$ are sufficiently generic at $\mathbf{p}$, and $\dim R_\gamma \geqslant m$, then $\overline{\operatorname{crit}(z_0,\dots,z_m)_{|R_\gamma}}$ is precisely the m-dimensional absolute polar variety of $\overline{R_\gamma}$ at $\mathbf{p}$.

Proposition 5.7. *For all m such that $0\leqslant m\leqslant n$, there is an equality of sets given by*

$$\Theta^m_{\mathbf{A}^\bullet,\mathbf{z}} = \bigcup_{0\leqslant j\leqslant m}\Big(\bigcup_{\mathbf{a}\in\mathbb{C}^{j+1}}\operatorname{supp}\big(\Phi^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{a})\big)\Big).$$

Proof. We proceed by induction on m.

When $m=0$, the claim is equivalent to

$$\eta\Big(|\operatorname{gecc}^\bullet(\mathbf{A}^\bullet)| \;\cap\; \operatorname{im} dz_0\Big) = \bigcup_{a_0\in\mathbb{C}}\operatorname{supp}(\phi_{z_0-a_0}[-1]\mathbf{A}^\bullet),$$

which follows from the first equality of Theorem 3.4.

Now suppose the claim is true for $m-1$, for all complexes and all choices of coordinates, where $1\leqslant m\leqslant n$. We wish to prove the claim for m.

Fix $a_0\in\mathbb{C}$. It suffices to prove that

$$(\dagger)\;\; V(z_0-a_0)\cap\Theta^m_{\mathbf{A}^\bullet,\mathbf{z}} = \operatorname{supp}(\phi_{z_0-a_0}[-1]\mathbf{A}^\bullet)\cup\bigcup_{1\leqslant j\leqslant m}\Big(\bigcup_{(a_1,\dots,a_j)\in\mathbb{C}^j}\operatorname{supp}\big(\Phi^j_{\mathbf{A}^\bullet,\mathbf{z}}(a_0,a_1,\dots,a_j)\big)\Big).$$

Fix $\mathbf{p}\in V(z_0-a_0)\cap Y$. We wish to show that $\mathbf{p}$ is in the set on the left side of (†) if and only if $\mathbf{p}$ is in the set on the right side of (†). There are two cases.

case 1: $\mathbf{p}\in\operatorname{supp}(\phi_{z_0-a_0}[-1]\mathbf{A}^\bullet)$.

This case follows trivially from the $m=0$ discussion above.

case 2: $\mathbf{p}\notin\operatorname{supp}(\phi_{z_0-a_0}[-1]\mathbf{A}^\bullet)$.

Recalling Proposition 5.2, we apply the inductive hypothesis to $\psi_{z_0-a_0}[-1]\mathbf{A}^\bullet$ and the rotated coordinates $\tilde{\mathbf{z}}$ to obtain that

$$\nu\Big(|\mathbb{P}(\mathrm{gecc}^\bullet(\psi_{z_0-a_0}[-1]\mathbf{A}^\bullet))| \ \cap\ (\mathcal{U}\times\{0\}\times\mathbb{P}^{m-1}\times\{\mathbf{0}\})\Big) \ =$$

$$\bigcup_{1\leqslant j\leqslant m}\Big(\bigcup_{(a_1,\dots,a_j)\in\mathbb{C}^j}\operatorname{supp}\big(\Phi^j_{\mathbf{A}^\bullet,\mathbf{z}}(a_0,a_1,\dots,a_j)\big)\Big).$$

By Theorem 3.2, $\mathrm{gecc}^\bullet(\psi_{z_0-a_0}[-1]\mathbf{A}^\bullet) \ = \ \big(T^*_{z_0,\mathbf{A}^\bullet}\mathcal{U}\big)^\bullet\odot(V(z_0-a_0)\times\mathbb{C}^{n+1})$. As we are in the case where $\mathbf{p}\not\in\operatorname{supp}\phi_{z_0-a_0}[-1]\mathbf{A}^\bullet$, Lemma 5.4 tells us that

$$\big|\big(T^*_{z_0,\mathbf{A}^\bullet}\mathcal{U}\big)^\bullet\big|_{\mathbf{p}} \ = \ |\,\mathrm{gecc}^\bullet(\mathbf{A}^\bullet)|_{\mathbf{p}} + \ < d_{\mathbf{p}}z_0 > .$$

It is now trivial to show that: $\mathbf{p}\in V(z_0-a_0)\cap\nu\Big(|\mathbb{P}(\mathrm{gecc}^\bullet(\mathbf{A}^\bullet))| \ \cap\ (\mathcal{U}\times\mathbb{P}^m\times\{\mathbf{0}\})\Big)$ if and only if

$$\mathbf{p}\in\nu\Big(|\mathbb{P}(\mathrm{gecc}^\bullet(\psi_{z_0-a_0}[-1]\mathbf{A}^\bullet))| \ \cap\ (\mathcal{U}\times\{0\}\times\mathbb{P}^{m-1}\times\{\mathbf{0}\})\Big),$$

if and only if

$$\mathbf{p}\in\bigcup_{1\leqslant j\leqslant m}\Big(\bigcup_{(a_1,\dots,a_j)\in\mathbb{C}^j}\operatorname{supp}\big(\Phi^j_{\mathbf{A}^\bullet,\mathbf{z}}(a_0,a_1,\dots,a_j)\big)\Big),$$

i.e., since we are in case 2, that $\mathbf{p}$ is in the set on the left side of (†) if and only if $\mathbf{p}$ is in the set on the right side of (†). □

Corollary 5.8. *For all m such that $0\leqslant m\leqslant n$, there is an equality of sets given by*

$$\Theta^m_{\mathbf{A}^\bullet,\mathbf{z}} = \bigcup_{0\leqslant j\leqslant m}\operatorname{supp}({}^\bullet\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}).$$

Proof. This is actually a point-set topology proof. Let $E^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{a})$ denote the set of those $\mathbf{x}\in Y$ such that the stalk of $\Phi^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{a})$ at $\mathbf{x}$ is not zero. Hence, $\operatorname{supp}(\Phi^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{a})) = \overline{E^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{a})}$, and $\operatorname{supp}({}^\bullet\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}) = \overline{\bigcup_{\mathbf{a}\in\mathbb{C}^{j+1}}E^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{a})}$.

Now, Proposition 5.7 tells us that $\displaystyle\bigcup_{0\leqslant j\leqslant m}\Big(\bigcup_{\mathbf{a}\in\mathbb{C}^{j+1}}\overline{E^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{a})}\Big)$ is closed. It is a trivial topology proof to show then that

$$\bigcup_{0\leqslant j\leqslant m}\Big(\bigcup_{\mathbf{a}\in\mathbb{C}^{j+1}}\overline{E^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{a})}\Big) \ = \ \bigcup_{0\leqslant j\leqslant m}\overline{\Big(\bigcup_{\mathbf{a}\in\mathbb{C}^{j+1}}E^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{a})\Big)}. \quad \square$$

Lemma 5.9. *Fix m. If, for all $j\leqslant m$, $\dim(\mathcal{W}\cap\Theta^j_{\mathbf{A}^\bullet,\mathbf{z}})\leqslant j$, then, for all $j\leqslant m$, $|\mathbb{P}(\mathrm{gecc}^\bullet(\mathbf{A}^\bullet))|$ properly intersects $\mathcal{W}\times\mathbb{P}^j\times\{\mathbf{0}\}$ inside $\mathcal{U}\times\mathbb{P}^n$.*

Moreover, whenever $|\mathbb{P}(\mathrm{gecc}^\bullet(\mathbf{A}^\bullet))|$ properly intersects $\mathcal{W}\times\mathbb{P}^j\times\{\mathbf{0}\}$ inside $\mathcal{U}\times\mathbb{P}^n$ for all $j\leqslant m$, there is an equality of sets, inside of $\mathcal{W}$, given by $\Theta^m_{\mathbf{A}^\bullet,\mathbf{z}} = \bigcup_{j\leqslant m}\Gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}$.

Finally, fix j, where $0 \leqslant j \leqslant n$. Suppose that the analytic set $|\mathbb{P}(\mathrm{gecc}^\bullet(\mathbf{A}^\bullet))|$ properly intersects $\mathcal{W} \times \mathbb{P}^j \times \{\mathbf{0}\}$ inside $\mathcal{U} \times \mathbb{P}^n$. Then, the graded enriched cycle $\mathbb{P}(\mathrm{gecc}^\bullet(\mathbf{A}^\bullet))$ properly intersects $\mathcal{W} \times \mathbb{P}^j \times \{\mathbf{0}\}$ inside $\mathcal{U} \times \mathbb{P}^n$, and, inside of $\mathcal{W}$,

$$\Gamma^j_{\mathbf{A}^\bullet,\mathbf{z}} = \left|\nu_*\Big(\mathbb{P}(\mathrm{gecc}^\bullet(\mathbf{A}^\bullet)) \odot \mathcal{W} \times \mathbb{P}^j \times \{\mathbf{0}\}\Big)\right|,$$

where ν_* denotes the proper push-forward.

Proof. Recall that we use $[w_0 : w_1 : \cdots : w_n]$ for homogeneous coordinates on $\mathbb{P}^n$.

Assume that, for all $j \leqslant m$, $\dim(\mathcal{W} \cap \Theta^j_{\mathbf{A}^\bullet,\mathbf{z}}) \leqslant j$, and suppose that E is a component of $|\mathbb{P}(\mathrm{gecc}^\bullet(\mathbf{A}^\bullet))| \cap (\mathcal{W} \times \mathbb{P}^j \times \{\mathbf{0}\})|$ such that $\dim E \geqslant j+1$, i.e., E is non-proper component of the intersection. As $\eta(E) \subseteq \Theta^j_{\mathbf{A}^\bullet,\mathbf{z}}$, $\dim(\eta(E)) \leqslant j$, by our hypothesis. Therefore, the generic fibre of E over $\eta(E)$ has dimension at least 1. Consequently, the generic fibre of E intersects the copy of $\mathbb{P}^{n-1}$ given by $V(w_j)$. Hence, $E \cap V(w_j) \subseteq |\mathbb{P}(\mathrm{gecc}^\bullet(\mathbf{A}^\bullet))| \cap (\mathcal{W} \times \mathbb{P}^{j-1} \times \{\mathbf{0}\})$ and $\dim(E \cap V(w_j)) \geqslant j$.

Thus, by induction, we arrive at the fact that $|\mathbb{P}(\mathrm{gecc}^\bullet(\mathbf{A}^\bullet))| \cap (\mathcal{W} \times \mathbb{P}^0 \times \{\mathbf{0}\})$ has a component with a generic fibre of dimension at least 1. This contradiction proves that, for all $j \leqslant m$, $|\mathbb{P}(\mathrm{gecc}^\bullet(\mathbf{A}^\bullet))|$ properly intersects $\mathcal{W} \times \mathbb{P}^j \times \{\mathbf{0}\}$ inside $\mathcal{U} \times \mathbb{P}^n$.

Now, assume that $|\mathbb{P}(\mathrm{gecc}^\bullet(\mathbf{A}^\bullet))|$ properly intersects $\mathcal{W} \times \mathbb{P}^j \times \{\mathbf{0}\}$ inside $\mathcal{U} \times \mathbb{P}^n$ for all $j \leqslant m$.

If $j \leqslant m$, then $\Theta^j_{\mathbf{A}^\bullet,\mathbf{z}} \subseteq \Theta^m_{\mathbf{A}^\bullet,\mathbf{z}}$. As $\Gamma^j_{\mathbf{A}^\bullet,\mathbf{z}} \subseteq \Theta^j_{\mathbf{A}^\bullet,\mathbf{z}}$, it follows that $\bigcup_{j\leqslant m} \Gamma^j_{\mathbf{A}^\bullet,\mathbf{z}} \subseteq \Theta^m_{\mathbf{A}^\bullet,\mathbf{z}}$. We need to show the reverse containment.

Suppose that C is an irreducible component of $\Theta^m_{\mathbf{A}^\bullet,\mathbf{z}}$ and that E is an irreducible component of $|\mathbb{P}(\mathrm{gecc}^\bullet(\mathbf{A}^\bullet))| \cap (\mathcal{W} \times \mathbb{P}^m \times \{\mathbf{0}\})$ such that $\eta(E) = C$. By hypothesis, $\dim E = m$, and so $\dim C \leqslant m$.

Suppose that $\dim C \leqslant m-1$, i.e., that C is a component of $\Theta^m_{\mathbf{A}^\bullet,\mathbf{z}}$, but not a component of $\Gamma^m_{\mathbf{A}^\bullet,\mathbf{z}}$. Then, the generic fibre of E must be at least 1-dimensional. Hence, $V(w_m)$ intersects the generic fibres of E, and so $C = \eta(E \cap V(w_m)) \subseteq \Theta^{m-1}_{\mathbf{A}^\bullet,\mathbf{z}}$. Proceeding inductively, we conclude that: if C is a j-dimensional irreducible component of $\Theta^m_{\mathbf{A}^\bullet,\mathbf{z}}$, then C is a component of $\Theta^j_{\mathbf{A}^\bullet,\mathbf{z}}$ and, therefore, a component of $\Gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}$. This proves that $\Theta^m_{\mathbf{A}^\bullet,\mathbf{z}} = \bigcup_{j\leqslant m} \Gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}$.

Finally, suppose, for a fixed j, that the analytic set $|\mathbb{P}(\mathrm{gecc}^\bullet(\mathbf{A}^\bullet))|$ properly intersects $\mathcal{W} \times \mathbb{P}^j \times \{\mathbf{0}\}$ inside $\mathcal{U} \times \mathbb{P}^n$.

As every component of $\mathbb{P}(\mathrm{gecc}^\bullet(\mathbf{A}^\bullet))$ is purely n-dimensional, the components of $\mathbb{P}(\mathrm{gecc}^\bullet(\mathbf{A}^\bullet))$ are the same as the components of the underlying set $|\mathbb{P}(\mathrm{gecc}^\bullet(\mathbf{A}^\bullet))|$. Thus, $\mathbb{P}(\mathrm{gecc}^\bullet(\mathbf{A}^\bullet))$ properly intersects $\mathcal{W} \times \mathbb{P}^j \times \{\mathbf{0}\}$ inside $\mathcal{U} \times \mathbb{P}^n$, and so the dimension of every component of $\mathbb{P}(\mathrm{gecc}^\bullet(\mathbf{A}^\bullet)) \odot \mathcal{W} \times \mathbb{P}^j \times \{\mathbf{0}\}$ is equal to j. Therefore, by definition of the proper push-forward, the components of $\left|\nu_*\Big(\mathbb{P}(\mathrm{gecc}^\bullet(\mathbf{A}^\bullet)) \odot \mathcal{W} \times \mathbb{P}^j \times \{\mathbf{0}\}\Big)\right|$ are precisely the j-dimensional components of $\nu\Big(|\mathbb{P}(\mathrm{gecc}^\bullet(\mathbf{A}^\bullet))| \cap (\mathcal{W} \times \mathbb{P}^j \times \{\mathbf{0}\})\Big)$. □

Theorem 5.10. *Let $\mathbf{p} \in Y$, and fix m such that $0 \leqslant m \leqslant n$. Then, the following are equivalent:*

a) for all j such that $0 \leqslant j \leqslant m$, $\dim_{\mathbf{p}} \mathrm{supp}(\Phi^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p})) \leqslant 0$;

b) for all j such that $0 \leqslant j \leqslant m$, $\dim_{\mathbf{p}} \Big(V(z_0 - p_0, \ldots, z_{j-1} - p_{j-1}) \cap \mathrm{supp}({}^\bullet\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}})\Big) \leqslant 0$;

c) for all j such that $0 \leqslant j \leqslant m$, $\dim_{\mathbf{p}} \Big(V(z_0 - p_0, \ldots, z_{j-1} - p_{j-1}) \cap \Theta^j_{\mathbf{A}^\bullet,\mathbf{z}}\Big) \leqslant 0$.

d) for all j such that $0 \leqslant j \leqslant m$, there exists an open neighborhood $\mathcal{W}$ of $\mathbf{p}$ in $\mathcal{U}$ such that $\big|\mathbb{P}\big(\operatorname{gecc}^\bullet(\mathbf{A}^\bullet)\big)\big|$ properly intersects $\mathcal{W} \times \mathbb{P}^j \times \{\mathbf{0}\}$ inside $\mathcal{W} \times \mathbb{P}^n$ and

$$\dim_{\mathbf{p}} \Big(V(z_0 - p_0, \dots, z_{j-1} - p_{j-1}) \cap \big|\nu_*\big(\mathbb{P}\big(\operatorname{gecc}^\bullet(\mathbf{A}^\bullet)\big) \ \odot \ \mathcal{W} \times \mathbb{P}^j \times \{\mathbf{0}\}\big)\big|\Big) \leqslant 0.$$

Proof. Given the results of 5.7, 5.8, and 5.9, this is an easy exercise; we leave it to the reader. □

Definition 5.11. The coordinates $\mathbf{z} = (z_0, \dots, z_n)$ are $\mathbf{A}^\bullet$-*isolating at* $\mathbf{p}$ if and only if the equivalent conditions of Theorem 5.10 hold for $m = d - 1$.

Here, when $d \leqslant 0$, we mean that there is no condition on the coordinates.

Note that this is a condition on only the first d coordinates, and occasionally we will simply say that $(z_0, \dots, z_{d-1})$ are $\mathbf{A}^\bullet$-isolating at $\mathbf{p}$. Note, also, that if $\mathbf{z}$ is $\mathbf{A}^\bullet$-isolating at $\mathbf{p}$, then 5.10.c implies that, for all j such that $0 \leqslant j \leqslant d-1$, $\dim_{\mathbf{p}} \Theta^j_{\mathbf{A}^\bullet, \mathbf{z}} \leqslant j$.

If $Z \subseteq Y$, then we say that the coordinates $\mathbf{z}$ are $\mathbf{A}^\bullet$-*isolating on* Z if and only if, for all $\mathbf{x} \in Z$, $\mathbf{z}$ is $\mathbf{A}^\bullet$-isolating at $\mathbf{x}$.

Remark 5.12. We could use 5.10.a to define a seemingly more general notion, that of an $\mathbf{A}^\bullet$-*isolating sequence of arbitrary functions* $(g_0, \dots, g_n)$, or perhaps a better term would be an $\mathbf{A}^\bullet$-*regular sequence*.

However, this notion can be easily recovered from our current set-up by considering the graph map $G : \mathcal{U} \to \mathbb{C}^{n+1} \times \mathcal{U}$, given by $G(\mathbf{x}) = (g_0(\mathbf{x}), \dots, g_n(\mathbf{x}), \mathbf{x})$. Use $(u_0, \dots, u_n, z_0, \dots, z_n)$ as coordinates on $\mathbb{C}^{n+1} \times \mathcal{U}$. Then, the sequence $(g_0, \dots, g_n)$ is $\mathbf{A}^\bullet$-isolating if and only if $(u_0, \dots, u_n)$ is $RG_*\mathbf{A}^\bullet$-isolating.

Some immediate properties of $\mathbf{A}^\bullet$-isolating coordinate are:

Proposition 5.13. *If $\mathcal{R}$ is Whitney a) and $\mathbf{z}$ is essentially transverse to $\mathcal{R}$ at $\mathbf{p}$, then $\mathbf{z}$ is $\mathbf{A}^\bullet$-isolating at $\mathbf{p}$. In particular, $\mathbf{A}^\bullet$-isolating coordinates are generic.*

If $\mathbf{z}$ is $\mathbf{A}^\bullet$-isolating at $\mathbf{p}$, there exists an open neighborhood of $\mathbf{p}$ on which $\mathbf{z}$ is $\mathbf{A}^\bullet$-isolating.

The coordinates $\mathbf{z}$ are $\mathbf{A}^\bullet$-isolating if and only if $\mathbf{z}$ are ${}^\mu H^k(\mathbf{A}^\bullet)$-isolating for all k, and, in this case, for all k and for all j, $0 \leqslant j \leqslant n$,

$${}^k\gamma^j_{\mathbf{A}^\bullet, \mathbf{z}}(\mathbf{x}) = {}^0\gamma^j_{{}^\mu H^k(\mathbf{A}^\bullet), \mathbf{z}}(\mathbf{x}).$$

Proof. The first claim follows from Corollary 4.4. In the second claim, the existence of the open neighborhood on which $\mathbf{z}$ is $\mathbf{A}^\bullet$-isolating follows from the fact that that fibre dimension of $(z_0, \dots, z_{j-1})_{|_{\Theta^j_{\mathbf{A}^\bullet, \mathbf{z}}}}$ is upper-semicontinuous.

The remainder of the proposition is immediate from the characterization of isolating given in 5.10.a together with the well-known facts that ${}^\mu H^k$ commutes with the shifted nearby and vanishing cycles, and that

$$\operatorname{supp} \mathbf{A}^\bullet = \bigcup_k \operatorname{supp} {}^\mu H^k(\mathbf{A}^\bullet). \quad \square$$

Proposition 5.14. *Suppose that the coordinates* $\mathbf{z}$ *are* $\mathbf{A}^\bullet$*-isolating at* $\mathbf{p}$.

Then, there exists an open neighborhood $\mathcal{W}$ *of* $\mathbf{p}$ *such that, for* $j \leqslant d-1$,

$$(\mathcal{W} - \{\mathbf{p}\}) \cap \operatorname{supp}\big(\psi_{z_j-p_j}[-1]\dots\psi_{z_0-p_0}[-1]\mathbf{A}^\bullet\big) = (\mathcal{W} - \{\mathbf{p}\}) \cap V(z_0-p_0,\dots,z_j-p_j) \cap \operatorname{supp}\mathbf{A}^\bullet,$$

and the dimension of these sets is at most $d-j-1$*, where, as usual, the dimension of the empty set is taken to be* $-\infty$.

Therefore,

$$\dim_{\mathbf{p}}\big(V(z_0-p_0,\dots,z_{d-1}-p_{d-1}) \cap \operatorname{supp}\mathbf{A}^\bullet\big) \leqslant 0,$$

and, for all j *such that* $0 \leqslant j \leqslant n$,

$$\dim_{\mathbf{p}}(\operatorname{supp}\psi_{z_j-p_j}[-1]\dots\psi_{z_0-p_0}[-1]\mathbf{A}^\bullet) \leqslant d-j-1;$$

in particular, if $j \geqslant d$*, then* $H^\bullet\big(\psi_{z_j-p_j}[-1]\dots\psi_{z_0-p_0}[-1]\mathbf{A}^\bullet\big)_{\mathbf{p}} = 0$.

Proof. Using the characterization of $\mathbf{A}^\bullet$-isolating given in 5.10.a, we conclude that there exists an open neighborhood $\mathcal{W}$ of $\mathbf{p}$ such that, for all $j \leqslant d-1$, $(\mathcal{W}-\{\mathbf{p}\}) \cap \operatorname{supp}\big(\Phi^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p})\big) = \emptyset$. Applying Corollary 3.6 inductively, we arrive at the support equality of the proposition, together with the dimension statement.

If we let $j = d-1$ in the support equality, we conclude that

$$\dim_{\mathbf{p}}\big(V(z_0-p_0,\dots,z_{d-1}-p_{d-1}) \cap \operatorname{supp}\mathbf{A}^\bullet\big) \leqslant 0.$$

Moreover, the support equality and its accompanying dimension statement immediately imply that, for all j such that $0 \leqslant j \leqslant d-1$,

$$\dim_{\mathbf{p}}(\operatorname{supp}\psi_{z_j-p_j}[-1]\dots\psi_{z_0-p_0}[-1]\mathbf{A}^\bullet) \leqslant d-j-1;$$

in particular,

$$\dim_{\mathbf{p}}(\operatorname{supp}\psi_{z_{d-1}-p_{d-1}}[-1]\dots\psi_{z_0-p_0}[-1]\mathbf{A}^\bullet) \leqslant 0.$$

However, this certainly implies that $\mathbf{p} \notin \operatorname{supp}\psi_{z_d-p_d}[-1]\dots\psi_{z_0-p_0}[-1]\mathbf{A}^\bullet$, and the remainder of the proposition follows. □

The definition of $\mathbf{A}^\bullet$-isolating coordinates uses only the coordinate functions z_0 through z_{d-1}. On the other hand, Definition 5.1 uses the coordinate z_d in the definition of ${}^k\gamma^d_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p})$ and also gives the definition of ${}^k\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p})$ for $j > d$. However, the following corollary tells us that, if our coordinates are $\mathbf{A}^\bullet$-isolating, then ${}^k\gamma^d_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p})$ only depends on the coordinates $(z_0,\dots,z_{d-1})$ and that, for $j > d$, ${}^k\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p})$ is trivial.

Corollary 5.15. *If the coordinates* $\mathbf{z}$ *are* $\mathbf{A}^\bullet$*-isolating at* $\mathbf{p}$*, then*

$${}^k\gamma^d_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p}) \;\cong\; H^k\big(\psi_{z_{d-1}-p_{d-1}}[-1]\psi_{z_{d-2}-p_{d-2}}[-1]\dots\psi_{z_0-p_0}[-1]\mathbf{A}^\bullet\big)_{\mathbf{p}};$$

and, for $j > d$, ${}^k\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p}) \;:=\; 0$.

Proof. Let $\mathbf{A}^\bullet := \psi_{z_{d-1}-p_{d-1}}[-1]\psi_{z_{d-2}-p_{d-2}}[-1]\dots\psi_{z_0-p_0}[-1]\mathbf{A}^\bullet$. By Proposition 5.14, $\mathbf{p}$ is either an isolated point of the support of $\mathbf{A}^\bullet$ or is not in the support at all. In either case, the nearby cycles $\psi_{z_d-p_d}[-1]\mathbf{A}^\bullet$ are zero in a neighborhood of $\mathbf{p}$; therefore, ${}^k\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p}) = 0$ for $j > d$.

As for ${}^k\gamma^d_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p})$, consider the fundamental distinguished triangle

$$\mathbf{A}^\bullet_{|_{V(z_d-p_d)}}[-1] \;\to\; \psi_{z_d-p_d}[-1]\mathbf{A}^\bullet \;\to\; \phi_{z_d-p_d}[-1]\mathbf{A}^\bullet \xrightarrow{[1]} \mathbf{A}^\bullet_{|_{V(z_d-p_d)}}[-1].$$

As $\psi_{z_d-p_d}[-1]\mathbf{A}^\bullet = 0$ near $\mathbf{p}$, it follows immediately that $H^k(\mathbf{A}^\bullet)_\mathbf{p} \cong H^k(\phi_{z_d-p_d}[-1]\mathbf{A}^\bullet)_\mathbf{p}$, i.e., that

$$H^k\big(\psi_{z_{d-1}-p_{d-1}}[-1]\psi_{z_{d-2}-p_{d-2}}[-1]\dots\psi_{z_0-p_0}[-1]\mathbf{A}^\bullet\big)_\mathbf{p} = {}^k\gamma^d_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p}). \quad \square$$

Remark 5.16. As we mentioned before, being $\mathbf{A}^\bullet$-isolating is a condition on only the first d coordinates. Corollary 5.15 tells us that when the coordinates are $\mathbf{A}^\bullet$-isolating, then the characteristic polar modules also depend on only the first d coordinates.

On the other hand, it is useful to have a characterization of $\mathbf{A}^\bullet$-isolating that does not explicitly use the local dimension d of the support of $\mathbf{A}^\bullet$. Recall from Remark 5.6 that, as germs at $\mathbf{p}$, $\Theta^m_{\mathbf{A}^\bullet,\mathbf{z}} = \operatorname{supp}\mathbf{A}^\bullet$ for all $m \geqslant d$. At the same time, Proposition 5.14 tells us that, if $\mathbf{z}$ is $\mathbf{A}^\bullet$-isolating at $\mathbf{p}$, then

$$\dim_\mathbf{p}\big(V(z_0-p_0,\dots,z_{d-1}-p_{d-1}) \cap \operatorname{supp}\mathbf{A}^\bullet\big) \leqslant 0.$$

From these facts, it follows immediately that, if $\mathbf{z}$ is $\mathbf{A}^\bullet$-isolating at $\mathbf{p}$, then

$$\dim_\mathbf{p}\Big(V(z_0-p_0,\dots,z_{j-1}-p_{j-1}) \cap \Theta^j_{\mathbf{A}^\bullet,\mathbf{z}}\Big) \leqslant 0$$

for all j such that $d \leqslant j \leqslant n$.

What this means is that, while the characterization of $\mathbf{A}^\bullet$-isolating given by 5.10.c is a condition for all j such that $0 \leqslant j \leqslant d-1$, it is equivalent to define $\mathbf{A}^\bullet$-isolating by the seemingly stronger condition that, for all j such that $0 \leqslant j \leqslant n$,

$$\dim_\mathbf{p}\Big(V(z_0-p_0,\dots,z_{j-1}-p_{j-1}) \cap \Theta^j_{\mathbf{A}^\bullet,\mathbf{z}}\Big) \leqslant 0.$$

Corollary 5.17. *If R is a domain and $\mathbf{z}$ is $\mathbf{A}^\bullet$-isolating, then the Euler characteristic of the stalk of $\mathbf{A}^\bullet$ at $\mathbf{p}$ is given by*

$$\chi(\mathbf{A}^\bullet)_\mathbf{p} = \sum_{0\leqslant j\leqslant d}(-1)^j\left({}^{\mathrm{ord}}\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{x})\right) = \sum_{j,k}(-1)^{j+k}\operatorname{rk}\left({}^k\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{x})\right).$$

Proof. This follows immediately from Proposition 5.3, using $m = d$, together with Proposition 5.14. $\square$

Theorem 5.18. *If $\mathbf{z}$ is $\mathbf{A}^\bullet$-isolating at $\mathbf{p}$ and $\mathbf{A}^\bullet$ is a perverse sheaf on Y, then, for all j, ${}^\bullet\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p})$, is supported only in degree 0, i.e., ${}^k\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p}) = 0$ if $k \neq 0$.*

Moreover, whenever $\mathbf{z}$ *is* $\mathbf{A}^\bullet$*-isolating at* $\mathbf{p}$ *and all of the* ${}^\bullet\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p})$ *are supported only in degree* 0*, there is a complex of* R*-modules in which* ${}^0\gamma^{-m}_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p})$ *is placed in degree* m*,*

$$0 \to {}^0\gamma^d_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p}) \to {}^0\gamma^{d-1}_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p}) \to \cdots \to {}^0\gamma^1_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p}) \to {}^0\gamma^0_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p}) \to 0,$$

whose cohomology is isomorphic to $H^m(\mathbf{A}^\bullet)_\mathbf{p}$ *in degrees* $-d \leqslant m \leqslant 0$*; in addition, if* $m \geqslant 1$ *or* $m \leqslant -d-1$*, then* $H^m(\mathbf{A}^\bullet)_\mathbf{p} = 0$*.*

More generally, whenever $\mathbf{z}$ *is* $\mathbf{A}^\bullet$*-isolating at* $\mathbf{p}$*, for all* k*, there is a complex of* R*-modules in which* ${}^k\gamma^{-m}_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p})$ *is placed in degree* m*,*

$$0 \to {}^k\gamma^d_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p}) \to {}^k\gamma^{d-1}_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p}) \to \cdots \to {}^k\gamma^1_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p}) \to {}^k\gamma^0_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p}) \to 0,$$

whose cohomology is isomorphic to $H^m\big({}^\mu H^k(\mathbf{A}^\bullet)\big)_\mathbf{p}$ *in degrees* $-d \leqslant m \leqslant 0$*; in addition, if* $m \geqslant 1$ *or* $m \leqslant -d-1$*, then* $H^m\big({}^\mu H^k(\mathbf{A}^\bullet)\big)_\mathbf{p} = 0$*.*

Proof. The first statement follows at once from the facts that the functors $\psi_f[-1]$ and $\phi_f[-1]$ take perverse sheaves to perverse sheaves, and perverse sheaves that are supported at isolated points can have non-zero cohomology only in degree zero.

The last statement follows at once from the first and second, together with Proposition 5.13.

We must now prove the second statement. It will be convenient to adopt the convention that, for $j = 0$,

$$\psi_{z_{j-1}-p_{j-1}}[-1]\dots\psi_{z_0-p_0}[-1]\mathbf{A}^\bullet = \mathbf{A}^\bullet.$$

There are fundamental distinguished triangles given by

$$(\dagger) \qquad \mathbf{A}^\bullet_{|_{V(z_j-p_j)}}[-1] \;\to\; \psi_{z_j-p_j}[-1]\mathbf{A}^\bullet \;\to\; \phi_{z_j-p_j}[-1]\mathbf{A}^\bullet \;\xrightarrow{[1]}\; \mathbf{A}^\bullet_{|_{V(z_j-p_j)}}[-1].$$

For any j such that $0 \leqslant j \leqslant d$, let $\mathbf{A}^\bullet_{j-1} := \psi_{z_{j-1}-p_{j-1}}[-1]\dots\psi_{z_0-p_0}[-1]\mathbf{A}^\bullet$. The fact that all of the ${}^\bullet\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p})$ are supported only in degree 0, combined with (†) yields that, for $m \neq 0, 1$,

$$(*) \qquad H^{m-1}(\mathbf{A}^\bullet_{j-1})_\mathbf{p} = H^m(\mathbf{A}^\bullet_{j-1}[-1])_\mathbf{p} \;\cong H^m(\psi_{z_j-p_j}[-1]\mathbf{A}^\bullet_{j-1})_\mathbf{p} \;=\; H^m(\mathbf{A}^\bullet_j)_\mathbf{p},$$

and an exact sequence

$$(\ddagger) \qquad 0 \to H^{-1}\big(\mathbf{A}^\bullet_{j-1}\big)_\mathbf{p} \to H^0(\mathbf{A}^\bullet_j)_\mathbf{p} \to {}^0\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p}) \to H^0\big(\mathbf{A}^\bullet_{j-1}\big)_\mathbf{p} \to H^1(\mathbf{A}^\bullet_j)_\mathbf{p} \to 0.$$

An inductive application of (∗) implies that if $m \geqslant 1$ or if $m \leqslant -d-1$, then

$$H^m(\mathbf{A}^\bullet)_\mathbf{p} \;\cong\; H^{m+d}\big(\mathbf{A}^\bullet_{d-1}\big)_\mathbf{p},$$

and Corollary 5.15 tells us that this is isomorphic to ${}^{m+d}\gamma^d_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p})$. Now, our hypothesis that ${}^\bullet\gamma^d_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p})$ is concentrated in degree 0 implies that ${}^{m+d}\gamma^d_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p}) = 0$; this proves the last part of the proposition.

Another inductive application of (∗) implies that

$$H^1\big(\mathbf{A}^\bullet_j\big)_\mathbf{p} \;\cong\; {}^{d-j}\gamma^d_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p}) = 0,$$

where the last equality follows from our hypotheses if $d \neq j$, and if $d = j$, $H^1\big(\mathbf{A}^\bullet_j\big)_{\mathbf{p}} = 0$ by the last part of Proposition 5.14 . Thus, the exact sequences given by (‡) reduce to

$$0 \to H^{-1}\big(\mathbf{A}^\bullet_{j-1}\big)_{\mathbf{p}} \xrightarrow{a_j} H^0(\mathbf{A}^\bullet_j)_{\mathbf{p}} \xrightarrow{b_j} {}^0\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p}) \xrightarrow{c_j} H^0\big(\mathbf{A}^\bullet_{j-1}\big)_{\mathbf{p}} \to 0.$$

Therefore, it is immediate that there is a complex given by

$$0 \to {}^0\gamma^d_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p}) \xrightarrow{b_{d-1}\circ c_d} {}^0\gamma^{d-1}_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p}) \xrightarrow{b_{d-2}\circ c_{d-1}} \dots \xrightarrow{b_1\circ c_2} {}^0\gamma^1_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p}) \xrightarrow{b_0\circ c_1} {}^0\gamma^0_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p}) \to 0,$$

where ${}^0\gamma^{-m}_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p})$ stands in degree m, and the cohomology of this complex in degree $m \leqslant -1$ is given by

$$\frac{\ker(b_{m-1}\circ c_m)}{\operatorname{im}(b_m\circ c_{m+1})} = \frac{\ker(b_{m-1}\circ c_m)}{\operatorname{im}(b_m)} = \frac{\ker(b_{m-1}\circ c_m)}{\ker(c_m)} \cong \ker(b_{m-1}) = \operatorname{im}(a_{m-1}) \cong H^{-1}\big(\mathbf{A}^\bullet_{m-2}\big)_{\mathbf{p}}.$$

By another inductive application of (∗), this last module is isomorphic to $H^{-m}(\mathbf{A}^\bullet)_{\mathbf{p}}$. The remaining case, in which $m = 0$, is trivial. □

Definition 5.19. If $\mathbf{z}$ is $\mathbf{A}^\bullet$-isolating at $\mathbf{p}$, then we refer to the second chain complex of Theorem 5.18 as the *degree k Zawatsky complex (or the degree k Z-complex) of* $\mathbf{A}^\bullet$ *at* $\mathbf{p}$ *with respect to* $\mathbf{z}$.

Definition 5.20. If $\mathbf{z}$ is $\mathbf{A}^\bullet$-isolating on the open set $\mathcal{W}$, then, in light of 5.10.d, we may define the *graded, enriched, j-dimensional characteristic polar cycle of* $\mathbf{A}^\bullet$ *with respect to* $\mathbf{z}$, ${}^\bullet\Gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}$, *inside* $\mathcal{W}$ by

$${}^k\Gamma^j_{\mathbf{A}^\bullet,\mathbf{z}} := \nu_*\Big(\mathbb{P}(\mathrm{gecc}^k(\mathbf{A}^\bullet)) \ \odot\ \mathcal{W}\times\mathbb{P}^j\times\{\mathbf{0}\}\Big).$$

If $\mathbf{z}$ is $\mathbf{A}^\bullet$-isolating at $\mathbf{p}$, then there exists an open neighborhood $\mathcal{W}$ of $\mathbf{p}$ on which $\mathbf{z}$ is $\mathbf{A}^\bullet$-isolating; thus, the *germ of* ${}^k\Gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}$ *at* $\mathbf{p}$ is well-defined.

Remark 5.21. Note that, if $\mathbf{z}$ is $\mathbf{A}^\bullet$-isolating at $\mathbf{p}$, then, for all j, there is an equality of germs of sets: $\Gamma^j_{\mathbf{A}^\bullet,\mathbf{z}} = |{}^\bullet\Gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}|$.

Lemma 5.22. *Suppose that* $\mathbf{z}$ *is* $\mathbf{A}^\bullet$*-isolating at* $\mathbf{p}$. *Let* $\tilde{\mathbf{z}}$ *denote the rotated coordinate system* $(z_1, z_2, \dots, z_n, z_0)$. *Then,* $\tilde{\mathbf{z}}$ *is* $\big(\psi_{z_0-p_0}[-1]\mathbf{A}^\bullet\big)$*-isolating at* $\mathbf{p}$, *and for all j such that* $1 \leqslant j \leqslant n$, *for all k, there is an equality of germs of graded enriched cycles given by*

$${}^k\Gamma^{j-1}_{\psi_{z_0-p_0}[-1]\mathbf{A}^\bullet,\tilde{\mathbf{z}}} \ = \ {}^k\Gamma^j_{\mathbf{A}^\bullet,\mathbf{z}} \odot V(z_0-p_0).$$

Proof. That $\tilde{\mathbf{z}}$ is $\big(\psi_{z_0-p_0}[-1]\mathbf{A}^\bullet\big)$-isolating at $\mathbf{p}$ is immediate from the definition, once we show that $\dim_{\mathbf{p}}\operatorname{supp}(\psi_{z_0-p_0}[-1]\mathbf{A}^\bullet) \leqslant -1 + \dim_{\mathbf{p}}\operatorname{supp}\mathbf{A}^\bullet$.

If $\mathbf{p}\notin\operatorname{supp}\mathbf{A}^\bullet$, then $\dim_{\mathbf{p}}\operatorname{supp}(\psi_{z_0-p_0}[-1]\mathbf{A}^\bullet) = \dim_{\mathbf{p}}\operatorname{supp}\mathbf{A}^\bullet = -\infty$. If $\dim_{\mathbf{p}}\operatorname{supp}\mathbf{A}^\bullet = 0$, then $\mathbf{p}\notin\operatorname{supp}(\psi_{z_0-p_0}[-1]\mathbf{A}^\bullet)$, i.e., $\dim_{\mathbf{p}}\operatorname{supp}(\psi_{z_0-p_0}[-1]\mathbf{A}^\bullet) = -\infty$. Therefore, throughout the remainder of the proof, we suppose that $\dim_{\mathbf{p}}\operatorname{supp}\mathbf{A}^\bullet \geqslant 1$.

Since $\mathbf{z}$ is $\mathbf{A}^\bullet$-isolating at $\mathbf{p}$, $\dim_{\mathbf{p}} \operatorname{supp}(\phi_{z_0-p_0}[-1]\mathbf{A}^\bullet) \leqslant 0$, and so there exists an open neighborhood $\mathcal{W}$ of $\mathbf{p}$ such that $\phi_{z_0-p_0}[-1]\mathbf{A}^\bullet$ is zero when restricted to $\mathcal{W} - \{\mathbf{p}\}$. That

$$\dim_{\mathbf{p}} \operatorname{supp}(\psi_{z_0-p_0}[-1]\mathbf{A}^\bullet) \leqslant -1 + \dim_{\mathbf{p}} \operatorname{supp} \mathbf{A}^\bullet$$

now follows at once from Corollary 3.6.

Throughout the remainder of the proof, we shall work in the neighborhood $\mathcal{W}$ from above.

We have equalities of enriched cycles:

$${}^k\Gamma^{j-1}_{\psi_{z_0-p_0}[-1]\mathbf{A}^\bullet,\tilde{\mathbf{z}}} \;=\; \nu_*\big(\mathbb{P}(\mathrm{gecc}^k(\psi_{z_0-p_0}[-1]\mathbf{A}^\bullet)) \;\odot\; (\mathcal{W}\times\{0\}\times\mathbb{P}^{j-1}\times\{\mathbf{0}\})\big),$$

which, by Theorem 3.2, is equal to

$$\nu_*\Big(\mathbb{P}\big((T^*_{z_0,\mathbf{A}^\bullet}\mathcal{U})^k\big)\big) \;\odot\; V(z_0-p_0) \;\odot\; (\mathcal{W}\times\{0\}\times\mathbb{P}^{j-1}\times\{\mathbf{0}\})\Big) =$$

$$V(z_0-p_0) \;\odot\; \nu_*\Big(\mathbb{P}\big((T^*_{z_0,\mathbf{A}^\bullet}\mathcal{U})^k\big)\big) \;\odot\; (\mathcal{W}\times\{0\}\times\mathbb{P}^{j-1}\times\{\mathbf{0}\})\Big),$$

where the last equality follows from the projection formula. Thus, we would be finished if we could show that

$$(\dagger)\;\; \nu_*\Big(\mathbb{P}\big((T^*_{z_0,\mathbf{A}^\bullet}\mathcal{U})^k\big)\big) \;\odot\; V(w_0)\odot(\mathcal{W}\times\mathbb{P}^j\times\{\mathbf{0}\})\Big) \;=\; \nu_*\Big(\mathbb{P}(\mathrm{gecc}^k(\mathbf{A}^\bullet)) \;\odot\; (\mathcal{W}\times\mathbb{P}^j\times\{\mathbf{0}\})\Big).$$

As $j \geqslant 1$, both sides of (†) yield sets whose dimension at $\mathbf{p}$ is at least 1. Therefore, it suffices to prove (†) holds on $\mathcal{W}' := \mathcal{W} - \{\mathbf{p}\}$.

Suppose that $\mathrm{gecc}^k(\mathbf{A}^\bullet) = \sum_\beta E_\beta \big[\overline{T^*_{R_\beta}\mathcal{U}}\big]$. As $\dim_{\mathbf{p}} \phi_{z_0-p_0}[-1]\mathbf{A}^\bullet \leqslant 0$, we may apply Lemma 5.4 to conclude that, above $\mathcal{W}'$,

$$\mathbb{P}\big((T^*_{z_0,\mathbf{A}^\bullet}\mathcal{U})^k\big)\big) = \sum_\beta E_\beta \left[\mathbb{P}\big(\overline{T^*_{R_\beta}\mathcal{U}} + < dz_0 > \big)\right].$$

Thus, to prove (†), it suffices to show that, for $\mathbf{A}^\bullet$-visible strata R_β,

$$(\ddagger)\;\; \nu_*\Big(\mathbb{P}\big(\overline{T^*_{R_\beta}\mathcal{U}} + < dz_0 > \big)\big) \;\cdot\; V(w_0)\cdot(\mathcal{W}'\times\mathbb{P}^j\times\{\mathbf{0}\})\Big) \;=\; \nu_*\Big(\mathbb{P}(\overline{T^*_{R_\beta}\mathcal{U}}) \;\odot\; (\mathcal{W}'\times\mathbb{P}^j\times\{\mathbf{0}\})\Big).$$

However, this is easy.

Let $\tau : \mathcal{W}'\times(\mathbb{P}^n - \{[1:\mathbf{0}]\}) \to \mathcal{W}'\times\{0\}\times\mathbb{P}^{n-1}$ denote the projection, and note that, over $\mathcal{W}'$, Theorem 3.4 implies that $\mathbb{P}(\overline{T^*_{R_\beta}\mathcal{U}}) \subseteq \mathcal{W}'\times(\mathbb{P}^n - \{[1:\mathbf{0}]\})$. Now, one notes that $\tau_*\big(\mathbb{P}(\overline{T^*_{R_\beta}\mathcal{U}})\big)$ is precisely equal to the transverse intersection $\mathbb{P}\big(\overline{T^*_{R_\beta}\mathcal{U}} + < dz_0 > \big)\big) \;\cdot\; V(w_0)$, and then (‡) follows from the projection formula. □

Finally, we can prove the **Fundamental Theorem of Characteristic Polar Modules**:

Theorem 5.23. *If* $\mathbf{z}$ *is* $\mathbf{A}^\bullet$*-isolating at* $\mathbf{p}$ *, then for all* j *and* k*, where* $0 \leqslant j \leqslant d$*,* ${}^k\Gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}$ *properly intersects* $V(z_0-p_0,\dots,z_{j-1}-p_{j-1})$ *at* $\mathbf{p}$ *and*

$${}^k\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p}) \;\cong\; \big({}^k\Gamma^j_{\mathbf{A}^\bullet,\mathbf{z}} \;\odot\; V(z_0-p_0,\dots,z_{j-1}-p_{j-1})\big)_{\mathbf{p}},$$

where, when $j=0$, we mean that ${}^k\gamma^0_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p}) \;=\; \left({}^k\Gamma^0_{\mathbf{A}^\bullet,\mathbf{z}}\right)_{\mathbf{p}}$.

Proof. The proper intersection statement follows immediately from Theorem 5.10.d. The remainder of the proof is by induction on j.

When $j=0$, the claim reduces to

$$H^k(\phi_{z_0-p_0}[-1]\mathbf{A}^\bullet)_{\mathbf{p}} \cong \Big(\nu_*\big(\mathbb{P}(\mathrm{gecc}^k(\mathbf{A}^\bullet)) \;\odot\; \mathcal{W}\times\mathbb{P}^0\times\{\mathbf{0}\}\big)\Big)_{\mathbf{p}}.$$

This follows immediately from Theorem 3.3.

Now, suppose that $j \geqslant 1$ and that the theorem holds for $j-1$ (for arbitrary $\mathbf{A}^\bullet$ and $\mathbf{z}$). Then, by Proposition 5.2, our inductive hypothesis, and Lemma 5.22 (in that order),

$$ {}^k\gamma^j_{\mathbf{A}^\bullet,\mathbf{z}}(\mathbf{p}) = {}^k\gamma^{j-1}_{\psi_{z_0-x_0}[-1]\mathbf{A}^\bullet,\tilde{\mathbf{z}}}(\mathbf{p}) \;\cong\; \left({}^k\Gamma^{j-1}_{\mathbf{A}^\bullet,\tilde{\mathbf{z}}} \;\odot\; V(z_1-p_1,\dots,z_{j-1}-p_{j-1})\right)_{\mathbf{p}} \;\cong$$

$$\left({}^k\Gamma^{j}_{\mathbf{A}^\bullet,\mathbf{z}} \;\odot\; V(z_0-p_0) \;\odot\; V(z_1-p_1,\dots,z_{j-1}-p_{j-1})\right)_{\mathbf{p}} \;\cong$$

$$\left({}^k\Gamma^{j}_{\mathbf{A}^\bullet,\mathbf{z}} \;\odot\; V(z_0-p_0,\dots,z_{j-1}-p_{j-1})\right)_{\mathbf{p}}. \quad \square$$

Remark 5.24. It is useful at this point to step back from all of our definitions and notation, and to describe the result of Theorem 5.23 in terms of ordinary intersection theory. What we have proved throughout this section is the following.

The condition that, for all $j < \dim_{\mathbf{p}}\left(\operatorname{supp}\mathbf{A}^\bullet\right)$,

$$\dim_{\mathbf{p}}\operatorname{supp}\Big(\phi_{z_j-p_j}[-1]\psi_{z_{j-1}-p_{j-1}}[-1]\dots\psi_{z_0-p_0}[-1]\mathbf{A}^\bullet\Big) \leqslant 0$$

is equivalent to: there exists an open neighborhood $\mathcal{W}$ of $\mathbf{p}$ in $\mathcal{U}$ such that, for all strata R_β which have non-trivial normal modules (i.e., $\mathbb{H}^\bullet(\mathbb{N}_\beta,\mathbb{L}_\beta;\;\mathbf{A}^\bullet)\neq 0$), for all $j\leqslant n$, $\mathbb{P}(\overline{T^*_{R_\beta}\mathcal{U}})$ properly intersects $\mathcal{W}\times\mathbb{P}^j\times\{\mathbf{0}\}$ in $\mathcal{U}\times\mathbb{P}^n$, and

$$\dim_{\mathbf{p}}\Big\{V(z_0-p_0,\dots,z_{j-1}-p_{j-1}) \;\cap\; \nu_*\Big(\mathbb{P}(\overline{T^*_{R_\beta}\mathcal{U}}) \;\cdot\; \big(\mathcal{W}\times\mathbb{P}^j\times\{\mathbf{0}\}\big)\Big)\Big\} \leqslant 0.$$

Moreover, whenever these equivalent hold, for all $j\leqslant n$, for all k,

$$(\dagger)\qquad H^k\big(\phi_{z_j-p_j}[-1]\psi_{z_{j-1}-p_{j-1}}[-1]\dots\psi_{z_0-p_0}[-1]\mathbf{A}^\bullet\big)_{\mathbf{p}} \;\cong\; \bigoplus\big(H^{k-d_\beta}(\mathbb{N}_\beta,\mathbb{L}_\beta;\;\mathbf{A}^\bullet)\big)^{m_\beta},$$

where $d_\beta := \dim R_\beta$, the sum is over those β such that $d_\beta\geqslant j$ and R_β has non-trivial normal modules, and

$$m_\beta := \Big(V(z_0-p_0,\dots,z_{j-1}-p_{j-1}) \;\cdot\; \nu_*\Big(\mathbb{P}(\overline{T^*_{R_\beta}\mathcal{U}}) \;\cdot\; \big(\mathcal{W}\times\mathbb{P}^j\times\{\mathbf{0}\}\big)\Big)\Big)_{\mathbf{p}}.$$

If the coordinates $\mathbf{z}$ are generic enough at $\mathbf{p}$, and still assuming that $d_\beta\geqslant j$, then m_β is equal to the multiplicity of the j-dimensional absolute polar variety of $\overline{R_\beta}$ at $\mathbf{p}$. Note, however, while $\mathbf{A}^\bullet$-isolating coordinates at $\mathbf{p}$ are $\mathbf{A}^\bullet$-isolating throughout a neighborhood of $\mathbf{p}$, that – unless $\mathbf{p}$ is a non-singular point of $\overline{R_\beta}$ – one may **not** choose one set of coordinates which are generic enough to yield the polar multiplicities of $\overline{R_\beta}$ throughout a neighborhood of $\mathbf{p}$.

One interesting observation that one can make from looking at (†) is that, if all of the normal modules of strata are free (or merely torsion-free), then so are all of the characteristic polar modules at each point.

§6. **Lê-Vogel Cycles and Numbers**

We wish to apply the results of the previous section to the case where $Y := V(f)$ and $\mathbf{A}^\bullet := \phi_f[-1]\mathbf{F}^\bullet$. Corollary 5.17 and Theorem 5.18 tell us that the characteristic polar modules of $\phi_f[-1]\mathbf{F}^\bullet$ provide a great deal of information about the stalk cohomology of $\phi_f[-1]\mathbf{F}^\bullet$. Thus, what we need to do is to find an algebraic method for calculating $\nu_*\Big(\mathbb{P}(\mathrm{gecc}^k(\phi_f[-1]\mathbf{F}^\bullet)) \odot \mathcal{U}\times\mathbb{P}^j\times\{\mathbf{0}\}\Big)$. We also need to investigate when coordinates are $\phi_f[-1]\mathbf{F}^\bullet$-isolating.

Note that, if $f \equiv 0$, then $\phi_f[-1]\mathbf{F}^\bullet \cong \mathbf{F}^\bullet$, and therefore the relative results of this section also apply to the absolute situation of the previous section.

We recall the following definition from [**M1**].

Definition 6.1. The $\mathbf{F}^\bullet$*-critical locus of* f, $\Sigma_{\mathbf{F}^\bullet} f$, is equal to $\{\mathbf{x}\in X \mid H^\bullet(\phi_{f-f(\mathbf{x})}[-1]\mathbf{F}^\bullet)_{\mathbf{x}} \neq 0\}$. Hence, the closure, $\overline{\Sigma_{\mathbf{F}^\bullet} f} = \bigcup_v \mathrm{supp}(\phi_{f-v}[-1]\mathbf{F}^\bullet)$.

Proposition 6.2. *For all* $v \in \mathbb{C}$*,*

$$V(f-v)\cap\Sigma_{\mathbf{F}^\bullet} f \;=\; \eta(|\,\mathrm{gecc}^\bullet(\phi_{f-v}[-1]\mathbf{F}^\bullet)|) \;=\; \nu(|\mathbb{P}\big(\mathrm{gecc}^\bullet(\phi_{f-v}[-1]\mathbf{F}^\bullet)\big)|).$$

Proof. This follows immediately from the third equality of Proposition 2.5, applied to the complex $\phi_{f-v}[-1]\mathbf{F}^\bullet$. □

Note that, with our new notation, $d = \dim_{\mathbf{p}} \overline{\Sigma_{\mathbf{F}^\bullet} f}$.

At long last, we can define our generalizations of the Lê cycles and Lê numbers to analytic functions with arbitrarily singular domains.

Definition 6.3. For all j such that $0 \leqslant j \leqslant n$, for all k, we let

$${}^k\lambda^j_{f,\mathbf{z}}(\mathbf{p};\ \mathbf{F}^\bullet) \;:=\; {}^k\gamma^j_{\phi_f[-1]\mathbf{F}^\bullet,\mathbf{z}}(\mathbf{p});$$

that is

$${}^k\lambda^j_{f,\mathbf{z}}(\mathbf{p};\ \mathbf{F}^\bullet) \;:=\; H^k\big(\phi_{z_j-p_j}[-1]\psi_{z_{j-1}-p_{j-1}}[-1]\dots\psi_{z_0-p_0}[-1]\phi_f[-1]\mathbf{F}^\bullet\big)_{\mathbf{p}},$$

and we refer to ${}^k\lambda^j_{f,\mathbf{z}}(\mathbf{p};\ \mathbf{F}^\bullet)$ as the *degree* k*,* j*-dimensional Lê-Vogel (LêVo) module of* f *at* $\mathbf{p}$ *with respect to* $\mathbf{z}$ *with coefficients in* $\mathbf{F}^\bullet$.

For all j such that $0 \leqslant j \leqslant n$, we define $\Omega^j_{f,\mathbf{z}}(\mathbf{F}^\bullet)$ by

$$\Omega^j_{f,\mathbf{z}}(\mathbf{F}^\bullet) \ := \ \nu\Big(|\mathbb{P}(\mathrm{gecc}^\bullet(\phi_f[-1]\mathbf{F}^\bullet))| \ \cap \ (\mathcal{U}\times\mathbb{P}^j\times\{\mathbf{0}\})\Big),$$

and we define the *j-dimensional Lê-Vogel (Lê-Vogel) set of f with respect to* $\mathbf{z}$ *with coefficients in* $\mathbf{F}^\bullet$, $\Lambda^j_{f,\mathbf{z}}(\mathbf{F}^\bullet)$, to be the union of the j-dimensional components of $\Omega^j_{f,\mathbf{z}}(\mathbf{F}^\bullet)$.

If $\mathbf{z}$ are $\phi_f[-1]\mathbf{F}^\bullet$-isolating at $\mathbf{p}$, then we define the germ at $\mathbf{p}$ of the graded, enriched cycle ${}^\bullet\Lambda^j_{f,\mathbf{z}}(\mathbf{F}^\bullet)$ by

$${}^k\Lambda^j_{f,\mathbf{z}}(\mathbf{F}^\bullet) \ := \ {}^k\Gamma^j_{\phi_f[-1]\mathbf{F}^\bullet,\mathbf{z}},$$

and refer to this as the germ at $\mathbf{p}$ of the *degree k, j-dimensional Lê-Vogel (LêVo) cycle of f with respect to* $\mathbf{z}$ *with coefficients in* $\mathbf{F}^\bullet$.

Our earlier results allow us to quickly prove

Theorem 6.4. *For a generic choice of* $\mathbf{z}$*, the coordinates* $\mathbf{z}$ *are* $\phi_f[-1]\mathbf{F}^\bullet$*-isolating at* $\mathbf{p}$.

If the coordinates $\mathbf{z}$ *are* $\phi_f[-1]\mathbf{F}^\bullet$*-isolating at* $\mathbf{p}$*, then there is an open neighborhood,* $\mathcal{W}$*, of* $\mathbf{p}$ *on which* $\mathbf{z}$ *are* $\phi_f[-1]\mathbf{F}^\bullet$*-isolating.*

If the coordinates $\mathbf{z}$ *are* $\phi_f[-1]\mathbf{F}^\bullet$*-isolating at* $\mathbf{p}$*, then, for all* k*, for all* $j > \dim_{\mathbf{p}}\mathrm{supp}(\phi_f[-1]\mathbf{F}^\bullet)$*,* ${}^k\lambda^j_{f,\mathbf{z}}(\mathbf{p};\ \mathbf{F}^\bullet) = 0$.

The coordinates $\mathbf{z}$ *are* $\phi_f[-1]\mathbf{F}^\bullet$*-isolating on an open* $\mathcal{W}\subseteq\mathcal{U}$ *if and only if, for all* m *such that* $0 \leqslant m \leqslant n$*,* $|\mathbb{P}\big(\mathrm{gecc}^\bullet(\phi_f[-1]\mathbf{F}^\bullet)\big)|$ *properly intersects* $\mathcal{W}\times\mathbb{P}^m\times\{\mathbf{0}\}$ *in* $\mathcal{U}\times\mathbb{P}^n$ *and, for all* $\mathbf{x}\in\mathcal{W}$*,* $\dim_{\mathbf{x}}\Big(V(z_0-x_0,\dots,z_{m-1}-x_{m-1})\cap\Lambda^m_{f,\mathbf{z}}(\mathbf{F}^\bullet)\Big) \leqslant 0$.

If the coordinates $\mathbf{z}$ *are* $\phi_f[-1]\mathbf{F}^\bullet$*-isolating on an open* $\mathcal{W}\subseteq\mathcal{U}$*, then, on* $\mathcal{W}$*, there is an equality of graded, enriched cycles given by*

$${}^k\Lambda^j_{f,\mathbf{z}}(\mathbf{F}^\bullet) = \nu_*\Big(\mathbb{P}(\mathrm{gecc}^k(\phi_f[-1]\mathbf{F}^\bullet)) \ \odot \ \mathcal{W}\times\mathbb{P}^j\times\{\mathbf{0}\}\Big),$$

and, for all $\mathbf{x}\in V(f)$*, for all* k *and* j*, and isomorphism of modules*

$${}^k\lambda^j_{f,\mathbf{z}}(\mathbf{x};\ \mathbf{F}^\bullet) \ \cong \ \big({}^k\Lambda^j_{f,\mathbf{z}}(\mathbf{F}^\bullet) \ \odot \ V(z_0-x_0,\dots,z_{j-1}-x_{j-1})\big)_{\mathbf{x}}.$$

If the coordinates $\mathbf{z}$ *are* $\phi_f[-1]\mathbf{F}^\bullet$*-isolating at* $\mathbf{p}$*, then, for all* k*, there is a complex of* R*-modules in which* ${}^k\lambda^{-m}_{f,\mathbf{z}}(\mathbf{p};\ \mathbf{F}^\bullet)$ *is placed in degree* m*,*

$$0\to {}^k\lambda^d_{f,\mathbf{z}}(\mathbf{p};\ \mathbf{F}^\bullet)\to {}^k\lambda^{d-1}_{f,\mathbf{z}}(\mathbf{p};\ \mathbf{F}^\bullet)\to\cdots\to {}^k\lambda^1_{f,\mathbf{z}}(\mathbf{p};\ \mathbf{F}^\bullet)\to {}^k\lambda^0_{f,\mathbf{z}}(\mathbf{p};\ \mathbf{F}^\bullet)\to 0,$$

whose cohomology is isomorphic to $H^m\big({}^\mu H^k(\phi_f[-1]\mathbf{F}^\bullet)\big)_{\mathbf{p}} \cong H^m\big(\phi_f[-1]\big({}^\mu H^k(\mathbf{F}^\bullet)\big)\big)_{\mathbf{p}}$ *in degrees* $-d\leqslant m\leqslant 0$*; in addition, if* $m\geqslant 1$ *or* $m\leqslant -d-1$*, then*

$$H^m\big({}^\mu H^k(\phi_f[-1]\mathbf{F}^\bullet)\big)_{\mathbf{p}} \cong H^m\big(\phi_f[-1]\big({}^\mu H^k(\mathbf{F}^\bullet)\big)\big)_{\mathbf{p}} = 0.$$

If the base ring R is a domain, and the coordinates $\mathbf{z}$ are $\phi_f[-1]\mathbf{F}^\bullet$-isolating at $\mathbf{p}$, then

$$\chi(\phi_f[-1]\mathbf{F}^\bullet)_\mathbf{p} = \sum_{j,k}(-1)^{k+j}\,\mathrm{rk}\left({}^k\lambda^j_{f,\mathbf{z}}(\mathbf{p};\ \mathbf{F}^\bullet)\right).$$

Proof. All of the results here follow immediately by applying results of the previous section to our current situation. □

The first statement in Theorem 6.4 tells us that $\phi_f[-1]\mathbf{F}^\bullet$-isolating coordinates are generic, but we would like to have a reasonable idea of just how generic the coordinates need to be. Recall the definition of an $a_{f,\mathbf{F}^\bullet}$ partition from Definition 4.7.

Theorem 6.5. *Suppose that $\mathcal{S}$ is an $a_{f,\mathbf{F}^\bullet}$ partition of X, and let $\mathcal{S}' := \{S_\alpha \mid S_\alpha \subseteq V(f)\}$. Then, $\mathcal{S}'$ is a $\phi_f[-1]\mathbf{F}^\bullet$-partition of $V(f)$.*

Therefore, if $\mathbf{z}$ is essentially transverse to all of the $V(f)$ strata of an $a_{f,\mathbf{F}^\bullet}$ partition, then $\mathbf{z}$ is $\phi_f[-1]\mathbf{F}^\bullet$-isolating.

Proof. Suppose that $\mathbf{x} \in S_\beta \in \mathcal{S}'$, and that S_α is an $\mathbf{F}^\bullet$-visible stratum. Then, by Theorem 4.8, there is an inclusion of fibers over $\mathbf{x}$ given by

$$\pi\big(\mathrm{Ex}_{\mathrm{im}\, d\tilde f}(\overline{T^*_{S_\alpha}\mathcal{U}})\big)_\mathbf{x} \ \subseteq\ \mathbb{P}\big(T^*_{S_\beta}\mathcal{U}\big)_\mathbf{x}.$$

It follows that

$$\nu^{-1}(V(f))\ \cap\ \pi\big(\mathrm{Ex}_{\mathrm{im}\, d\tilde f}(\overline{T^*_{S_\alpha}\mathcal{U}})\big)\ \subseteq\ \bigcup_{S_\beta\in\mathcal{S}'}\mathbb{P}\big(T^*_{S_\beta}\mathcal{U}\big).$$

The proposition is now an immediate consequence of Theorem 3.4. □

Remark 6.6. Theorem 6.5 explains the importance of *good stratifications* and *prepolar coordinates* in our earlier work (see, for instance, [**M3**]) on Lê cycles and numbers. With our current notation and terminology, prepolar coordinates essentially transverse to all of the $V(f)$ strata of an $a_{f,\mathbb{C}^\bullet_\mathcal{U}}$ partition

The following is the **Fundamental Theorem of Lê-Vogel Cycles**.

Theorem 6.7. *Then, the following are equivalent:*

a) the coordinates $\mathbf{z}$ are $\phi_f[-1]\mathbf{F}^\bullet$-isolating at $\mathbf{p}$;

b) there exists a neighborhood $\mathcal{W}$ of $\mathbf{p}$ in $\mathcal{U}$ such that, for all j with $0 \leqslant j \leqslant n$, the graded, enriched cycle $\pi_*\big(\mathrm{Ex}_{\mathrm{im}\, d\tilde f}(\mathrm{gecc}^\bullet(\mathbf{F}^\bullet))\big)$ properly intersects $\mathcal{W}\times\mathbb{P}^j\times\{\mathbf{0}\}$ and

$$\dim_\mathbf{p} V(z_0-p_0,\dots,z_{j-1}-p_{j-1})\cap\left|\nu_*\Big(\pi_*\big(\mathrm{Ex}_{\mathrm{im}\, d\tilde f}(\mathrm{gecc}^\bullet(\mathbf{F}^\bullet))\big)\odot\mathcal{W}\times\mathbb{P}^j\times\{\mathbf{0}\}\Big)\right|\ \leqslant\ 0;$$

c) there exists a neighborhood $\mathcal{W}$ of $\mathbf{p}$ in $\mathcal{U}$ such that, for all j with $0 \leqslant j \leqslant n$, the graded, enriched cycle $\mathrm{Ex}_{\mathrm{im}\, d\tilde f}(\mathrm{gecc}^k(\mathbf{F}^\bullet))$ properly intersects $\mathcal{W}\times\mathbb{C}^{n+1}\times\mathbb{P}^j\times\{\mathbf{0}\}$ and

$$\dim_\mathbf{p} V(z_0-p_0,\dots,z_{j-1}-p_{j-1})\cap\left|\eta_*\Big(\tau_*\Big(\mathrm{Ex}_{\mathrm{im}\, d\tilde f}(\mathrm{gecc}^k(\mathbf{F}^\bullet))\ \odot\ \mathcal{U}\times\mathbb{C}^{n+1}\times\mathbb{P}^j\times\{\mathbf{0}\}\Big)\Big)\right|\ \leqslant\ 0.$$

Whenever the equivalent conditions above hold, there are equalities of germs at $\mathbf{p}$ *of enriched cycles given by*

$$^{k}\Lambda^{j}_{f,\mathbf{z}}(\mathbf{F}^\bullet) \ = \ \nu_*\Big(\pi_*\big(\mathrm{Ex}_{\mathrm{im}\, d\tilde f}(\mathrm{gecc}^k(\mathbf{F}^\bullet))\big) \ \odot \ \mathcal{U}\times\mathbb{P}^j\times\{\mathbf{0}\}\Big) =$$

$$\eta_*\Big(\tau_*\Big(\mathrm{Ex}_{\mathrm{im}\, d\tilde f}(\mathrm{gecc}^k(\mathbf{F}^\bullet)) \ \odot \ \mathcal{U}\times\mathbb{C}^{n+1}\times\mathbb{P}^j\times\{\mathbf{0}\}\Big)\Big),$$

where, in the last expression, η_* *induces an isomorphism onto its image, i.e., if we let*

$$\sum_V M_V[V] \ = \ \tau_*\Big(\mathrm{Ex}_{\mathrm{im}\, d\tilde f}(\mathrm{gecc}^k(\mathbf{F}^\bullet)) \ \odot \ \mathcal{U}\times\mathbb{C}^{n+1}\times\mathbb{P}^j\times\{\mathbf{0}\}\Big),$$

then, for all V *for which* $M_V\neq 0$, $\eta_{|_V}$ *is an isomorphism onto its image, and* $\eta_*(\sum_V M_V[V]) = \sum_V M_V[\eta(V)]$.

Proof. The equivalence of a) and b) follows immediately from 5.10. The equivalence of b) and c) follows trivially from the fact that $\nu\circ\pi = \eta\circ\tau$.

As for the formula, the first equality is immediate from Theorems 6.4 and 3.4.

Over $\mathrm{im}\, d\tilde f\times\mathbb{P}^n$, π has an inverse $\xi:\mathrm{im}\, d\tilde f\times\mathbb{P}^n\to\mathcal{U}\times\mathbb{C}^{n+1}\times\mathbb{P}^n$ given by $\xi(\mathbf{x},[\mathbf{u}]) = (\mathbf{x}, d_{\mathbf{x}}\tilde f, [\mathbf{u}])$. Therefore, elementary intersection theory implies that

$$\pi_*\big(\mathrm{Ex}_{\mathrm{im}\, d\tilde f}(\mathrm{gecc}^k(\mathbf{F}^\bullet))\big) \ \odot \ \mathcal{U}\times\mathbb{P}^j\times\{\mathbf{0}\} \ = \ \pi_*\Big(\mathrm{Ex}_{\mathrm{im}\, d\tilde f}(\mathrm{gecc}^k(\mathbf{F}^\bullet)) \ \odot \ \mathcal{U}\times\mathbb{C}^{n+1}\times\mathbb{P}^j\times\{\mathbf{0}\}\Big).$$

Now, the second equality follows from the fact that $\nu\circ\pi = \eta\circ\tau$.

The final isomorphism claim follows from the fact that

$$\tau\Big(\mathrm{Ex}_{\mathrm{im}\, d\tilde f}(\mathrm{gecc}^k(\mathbf{F}^\bullet)) \ \odot \ \mathcal{U}\times\mathbb{C}^{n+1}\times\mathbb{P}^j\times\{\mathbf{0}\}\Big)$$

lies inside $\mathrm{im}\, d\tilde f$, and – as with π in the above paragraph – η is invertible over $\mathrm{im}\, d\tilde f$. □

The importance of Theorem 6.7 is that the general theory of Vogel cycles (see [**G1**], [**G2**], [**V**], and [**M7**]) gives one an algorithm for calculating $\tau_*\Big(\mathrm{Ex}_{\mathrm{im}\, d\tilde f}(\mathrm{gecc}^k(\mathbf{F}^\bullet)) \odot \mathcal{U}\times\mathbb{C}^{n+1}\times\mathbb{P}^j\times\{\mathbf{0}\}\Big)$. This is the algorithm that we stated in a slightly more general form in Section 3 as Theorem 3.8. The corollary below follows immediately from 3.8; hence, we present this corollary without proof.

Corollary 6.8. *Suppose that* $\mathbf{z}$ *are* $\phi_f[-1]\mathbf{F}^\bullet$*-isolating at* $\mathbf{p}$*. Then, with respect to* $\mathbf{z}$*, the LêVo modules of* f *at* $\mathbf{p}$ *with coefficients in* $\mathbf{F}^\bullet$*, and the germs at* $\mathbf{p}$ *of the corresponding LêVo cycles may be calculated via the following process:*

We assume that we are working over a sufficiently small neighborhood of $\mathbf{p}$ *so that the coordinates are* $\phi_f[-1]\mathbf{F}^\bullet$*-isolating on the entire neighborhood.*

Let $\Pi^{n+1} := \mathrm{gecc}^k(\mathbf{F}^\bullet)$*. Then,* Π^{n+1} *properly intersects* $V\left(w_n - \dfrac{\partial\tilde f}{\partial z_n}\right)$*, and we may consider the enriched cycle defined by the intersection*

$$\sum_V M_V[V] \ := \ \Pi^{n+1}\odot V\left(w_n - \frac{\partial\tilde f}{\partial z_n}\right);$$

this enriched cycle may have some components contained in $\operatorname{im} d\tilde{f}$ *and some components not contained in* $\operatorname{im} d\tilde{f}$*. Let* $\Pi^n := \sum_{V \not\subseteq \operatorname{im} d\tilde{f}} M_V[V]$ *and let* $\Delta^n := \sum_{V \subseteq \operatorname{im} d\tilde{f}} M_V[V]$.

Now, proceed inductively: if we have Π^{j+1}*, then* $V\left(w_j - \dfrac{\partial \tilde{f}}{\partial z_j}\right)$ *properly intersects* Π^{j+1}*, and we define* Π^j *and* Δ^j *by the equality*

$$\Pi^{j+1} \odot V\left(w_j - \frac{\partial \tilde{f}}{\partial z_j}\right) \;=\; \Pi^j + \Delta^j,$$

where no component of Π^j *is contained in* $\operatorname{im} d\tilde{f}$*, and every component of* Δ^j *is contained in* $\operatorname{im} d\tilde{f}$*.*

Continue with this process until one obtains Π^0 *and* Δ^0*.*

Then, for all j*, as germs at* $\mathbf{p}$*,* ${}^k\Lambda^j_{f,\mathbf{z}}(\mathbf{F}^\bullet) = \eta_*(\Delta^j)$ *and*

$${}^k\lambda^j_{f,\mathbf{z}}(\mathbf{p};\ \mathbf{F}^\bullet) = V(z_0 - p_0, \dots, z_{j-1} - p_{j-1}) \odot \eta_*(\Delta^j).$$

Remark 6.9. It would, of course, be nice if we could use the process of Corollary 6.8 to decide whether or not the coordinates are $\phi_f[-1]\mathbf{F}^\bullet$-isolating. That is, one might hope that, if all of the intersections in Corollary 6.8 are proper – including the intersections of $V(z_0 - p_0, \dots, z_{j-1} - p_{j-1})$ and $\eta_*(\Delta^j)$ – then the coordinates are $\phi_f[-1]\mathbf{F}^\bullet$-isolating at $\mathbf{p}$. While we can not prove this in general, or find a counterexample, we do prove it below in the cases where $d \leqslant 2$.

In the general case, we must use the more unmanageable condition of requiring the coordinates to be essentially transverse to all of the $V(f)$ strata of an $a_{f,\mathbf{F}^\bullet}$ partition (recall Theorem 6.5).

First, we need an easy transversality lemma. For notational ease, we concentrate our attention at the origin.

Lemma 6.10. *Suppose that* Z *is an analytic subset of* $\mathcal{U}$*, and that* $\dim_{\mathbf{0}} V(z_0, \dots, z_j) \cap Z \leqslant 0$.
Then, for generic $\eta \in \mathbb{P}^j \times \{\mathbf{0}\}$*, there exists a open neighborhood* $\mathcal{W}$ *of* $\mathbf{0}$ *in* $\mathcal{U}$ *such that*

$$\mathbb{P}\big(\overline{T^*_{Z_{\rm reg}}\mathcal{U}}\big) \cap (\mathcal{W} \times \{\eta\}) \subseteq \{\mathbf{0}\} \times \mathbb{P}^n.$$

Proof. If $\mathbf{0} \notin Z$, then the result is trivial. So, assume that $\dim_{\mathbf{0}} V(z_0, \dots, z_j) \cap Z = 0$. Let $\xi : \mathbb{C}^{n+1} \to \mathbb{C}^{j+1}$ denote that projection onto the first $j+1$ coordinates.

As $\dim_{\mathbf{0}} V(z_0, \dots, z_j) \cap Z = 0$, there exists an open neighborhood $\mathcal{V}$ of $\mathbf{0}$ in $\mathcal{U}$, and an open neighborhood $\mathcal{Q}$ of $\mathbf{0}$ in $\mathbb{C}^{j+1}$ such that the restriction of ξ, $\tilde{\xi} : \mathcal{V} \cap Z \to \mathcal{Q}$, is a finite map. As $\tilde{\xi}$ is proper, we may Whitney stratify $\mathcal{V} \cap Z$ and $\operatorname{im} \tilde{\xi}$ in such a way that $\tilde{\xi}$ becomes a stratified map (see I.1.7 of [**G-M2**]).

Now, a generic hyperplane through the origin in $\mathbb{C}^{j+1}$ will transversely intersect all of the strata of $\operatorname{im} \tilde{\xi}$ near the origin, except possibly at the origin itself. It follows that, for generic $[a_0 : \cdots : a_j] \in \mathbb{P}^j$, $V(a_0 z_0 + \cdots + a_j z_j)$ transversely intersects all of the strata of $\mathcal{V} \cap Z$ near the origin, except perhaps at the origin itself. Fix such an $[a_0 : \cdots : a_j]$.

We proceed by contradiction. Suppose there were a parameterized analytic curve $\mathbf{q}(t) \in Z$ such that $\mathbf{q}(0) = \mathbf{0}$, $\mathbf{q}(t) \neq 0$ for $t \neq 0$, and

$$[a_0 dz_0 + \cdots + a_j dz_j] \in \left(\mathbb{P}(\overline{T^*_{Z_{\mathrm{reg}}}\mathcal{U}})\right)_{\mathbf{q}(t)}.$$

For t sufficiently small and unequal to 0, $\mathbf{q}(t)$ must be contained in one of the Whitney strata, M, of Z. By Whitney's condition a), we have that, for t small and unequal to 0,

$$(\dagger) \qquad [a_0 dz_0 + \cdots + a_j dz_j] \in \left(\mathbb{P}(T^*_M\mathcal{U})\right)_{\mathbf{q}(t)}.$$

As $\mathbf{q}'(t) \in T_{\mathbf{q}(t)}M$, (†) implies that $a_0 q_0'(t) + \cdots + a_j q_j'(t) \equiv 0$ for $t \neq 0$. As $\mathbf{q}(0) = \mathbf{0}$, it follows that $a_0 q_0(t) + \cdots + a_j q_j(t) \equiv 0$, i.e., that $\mathbf{q}(t) \in V(a_0 z_0 + \cdots + a_j z_j)$. However, this means that (†) contradicts the fact that $V(a_0 z_0 + \cdots + a_j z_j)$ transversely intersects M near the origin. □

The proof of the following proposition is not particularly difficult, but it does break up into a number of cases.

Proposition 6.11. *Assume that we are in the setting of Corollary 6.9, except that we do* **not** *assume that the coordinates* $\mathbf{z}$ *are* $\phi_f[-1]\mathbf{F}^\bullet$*-isolating at* $\mathbf{p}$*. Assume that* $d \leqslant 2$*.*

If there is a neighborhood of $\mathbf{p}$ *over which all of the intersections appearing in Corollary 6.9 are proper, including the intersections* $V(z_0 - p_0, \dots, z_{j-1} - p_{j-1}) \odot \eta_*(\Delta^j)$*, then the coordinates* $\mathbf{z}$ *are, in fact,* $\phi_f[-1]\mathbf{F}^\bullet$*-isolating at* $\mathbf{p}$*.*

Proof. We use the criterion in 5.10.c for showing that the coordinates $\mathbf{z}$ are $\phi_f[-1]\mathbf{F}^\bullet$-isolating at $\mathbf{p}$. For notational convenience, we will assume that $\mathbf{p} = \mathbf{0}$.

We use the notation from 6.8. Note that we have an equality of sets

$$|\operatorname{gecc}^\bullet(\mathbf{F}^\bullet)| \cap V\left(w_0 - \frac{\partial \tilde{f}}{\partial z_0}, \dots, w_n - \frac{\partial \tilde{f}}{\partial z_n}\right) = \bigcup_j |\Delta^j|;$$

this is easy to see, or follows rigorously from Proposition I.2.4 and Proposition I.2.15 of [**M7**]. Thus, applying η to both sides, and using the first part of Theorem 3.4, we conclude that, near $\mathbf{0}$,

$$\overline{\Sigma_{\mathbf{F}^\bullet} f} = \bigcup_j \eta(|\Delta^j|).$$

Therefore, our assumptions on proper intersections imply that

$$(*) \qquad \dim_{\mathbf{0}} V(z_0, z_1) \cap \overline{\Sigma_{\mathbf{F}^\bullet} f} \leqslant 0.$$

Let $S_\alpha \in \mathcal{S}$ be an $\mathbf{F}^\bullet$-visible stratum, and let us simply write E for the exceptional divisor $\operatorname{Ex}_{\operatorname{im} d\tilde{f}}(\overline{T^*_{S_\alpha}\mathcal{U}})$. Combining 5.10.c with Theorem 3.4, we see that what we need to show is that: for all j,

$$(\dagger) \qquad \dim_{\mathbf{0}} V(z_0, \dots, z_{j-1}) \cap \left|\nu\Big(\pi(E) \ \cap \ \big(\mathcal{U} \times \mathbb{P}^j \times \{\mathbf{0}\}\big)\Big)\right| \leqslant 0,$$

where Theorem 3.4 tells us that, as sets, $\pi(E)$ is a union of components of $\bigcup_\beta \mathbb{P}(\overline{T^*_{R_\beta}\mathcal{U}}) = \bigcup_\beta \mathbb{P}(T^*_{R_\beta}\mathcal{U})$ for some Whitney a) stratification, $\{R_\beta\}$, of $V(f)$. By $(*)$, the condition (†) holds for $j \geqslant 2$. We have only to show that (†) holds for $j = 0$ and $j = 1$.

We proceed by contradiction.

• Case 1: $j = 0$.

Suppose that there is a parameterized analytic curve $\mathbf{q}(t)$ such that $\mathbf{q}(0) = \mathbf{0}$, $\mathbf{q}(t) \neq \mathbf{0}$ for $t \neq 0$, and, for $t \neq 0$,

$$\mathbf{q}(t) \in \left| \nu\Big(\pi(E) \ \cap \ \big(\mathcal{U} \times \mathbb{P}^0 \times \{\mathbf{0}\}\big)\Big) \right|.$$

For t small and unequal to 0, there exists a single R_β such that $\mathbf{q}(t) \in R_\beta$. As $\mathbf{q}'(t) \in T_{\mathbf{q}(t)} R_\beta$ for small $t \neq 0$, it follows that $q_0'(t) \equiv 0$. Since $q_0(0) = 0$, we must have $q_0(t) \equiv 0$, i.e., $\mathbf{q}(t) \in V(z_0)$. Therefore, if we can show that (†) holds for $j = 1$, the $j = 0$ case will follow.

• Case 2: $j = 1$.

Suppose that C is a component of $\pi(E) \cap (\mathcal{U} \times \mathbb{P}^1 \times \{\mathbf{0}\})$ such that $\dim_{\mathbf{0}} V(z_0) \cap \nu(C) \geqslant 1$. Such a component C corresponds to a component $\widetilde{C}$ of $E \cap (\mathcal{U} \times \mathbb{C}^{n+1} \times \mathbb{P}^1 \times \{\mathbf{0}\})$, which must be contained in a component D of $\big(\mathrm{Bl}_{\mathrm{im}\, d\tilde{f}}(\overline{T^*_{S_\alpha}\mathcal{U}})\big) \cap (\mathcal{U} \times \mathbb{C}^{n+1} \times \mathbb{P}^1 \times \{\mathbf{0}\})$. Note that the dimension of D must be at least 2.

If $D \not\subseteq E$, then – looking at the definition of Π^2 in 6.8 – we find that $\tau(D) \subseteq \Pi^2$. Thus, if $D \not\subseteq E$, then

$$\tau(\widetilde{C}) \subseteq \tau(D \cap E) \subseteq \Pi^2 \cap V\left(\mathbf{w} - \frac{\partial \tilde{f}}{\partial \mathbf{z}}\right) = \Delta^1 \cup \Delta^0,$$

and so

$$V(z_0) \cap \nu(C) = V(z_0) \cap \nu(\pi(\widetilde{C})) \subseteq V(z_0) \cap \nu(\pi(D \cap E)) =$$
$$V(z_0) \cap \eta(\tau(D \cap E)) \subseteq V(z_0) \cap \big(\eta(\Delta^1) \cup \eta(\Delta^0)\big),$$

which, by assumption, has dimension of at most 0 at the origin. This is a contradiction.

It remains only for us to dispose of the case where $D \subseteq E$. Hence, we will be finished if we can show, for every component C of $\pi(E) \ \cap \ \big(\mathcal{U} \times \mathbb{P}^1 \times \{\mathbf{0}\}\big)$ of dimension at least 2, that $\dim_{\mathbf{0}} V(z_0) \cap \nu(C) \leqslant 0$.

We proceed by contradiction. Suppose that we have an irreducible C such that $\dim C \geqslant 2$ and such that $\dim_{\mathbf{0}} V(z_0) \cap \nu(C) \geqslant 1$.

If $\dim_{\mathbf{0}} \nu(C) = 0$, then there is nothing to show.

Suppose that $\dim_{\mathbf{0}} \nu(C) = 1$. As $\dim C \geqslant 2$, we must have that $C = \nu(C) \times \mathbb{P}^1 \times \{\mathbf{0}\}$. Let R_β be the stratum which contains $\nu(C) - \{\mathbf{0}\}$ in some neighborhood of the origin. Then, over a neighborhood of the origin,

$$\big(\nu(C) - \{\mathbf{0}\}\big) \times \mathbb{P}^1 \times \{\mathbf{0}\} \ \subseteq \ \mathbb{P}(T^*_{R_\beta}\mathcal{U}).$$

However, this means that, along the curve $\nu(C) - \{\mathbf{0}\}$, the tangent space to R_β is contained in $\{\mathbf{0}\} \times \mathbb{C}^{n-1}$. By parameterizing $\nu(C)$, we see that this implies that $\nu(C) \subseteq V(z_0, z_1)$ near the origin. This contradicts (∗).

Suppose, finally, that $\dim_{\mathbf{0}} \nu(C) = 2$. Then, there exists a 2-dimensional stratum R_β such that $C \subseteq \mathbb{P}(\overline{T^*_{R_\beta}\mathcal{U}})$, and $\nu(C) = \overline{R_\beta}$.

By $(*)$, $\overline{R_\beta} \not\subseteq V(z_0, z_1)$. Thus, for generic $[a_0 : a_1] \in \mathbb{P}^1$, $V(a_0z_0 + a_1z_1) \cap \overline{R_\beta}$ is 1-dimensional and $\big(V(a_0z_0 + a_1z_1) \cap \overline{R_\beta}\big) - \{\mathbf{0}\} \subseteq R_\beta$. Fix such an $[a_0 : a_1]$. Let $\mathbf{q}(t)$ be a parameterization of a component of $V(a_0z_0 + a_1z_1) \cap \nu(C)$ such that $\mathbf{q}(0) = \mathbf{0}$. Therefore, for all small $t \neq 0$, $\mathbf{q}'(t) \in T_{\mathbf{q}(t)}R_\beta$, $a_0q_0'(t)+a_1q_1'(t) = 0$, and $\mathbb{P}(T^*_{R_\beta}\mathcal{U})_{\mathbf{q}(t)}$ contains an element of the form $[b_0(t)dz_0+b_1(t)dz_1]$.

It follows that both $a_0q_0'(t) + a_1q_1'(t) = 0$ and $b_0(t)q_0'(t) + b_1(t)q_1'(t) = 0$ for all small $t \neq 0$; thus, either $q_0'(t) = q_1'(t) = 0$ or $[a_0 : a_1] = [b_0(t) : b_1(t)]$. If $q_0'(t) = q_1'(t) = 0$ for an infinite number of t values near $t = 0$, then we must have that both $q_0(t)$ and $q_1(t)$ are identically equal to zero; this would contradict $(*)$.

Thus, we must have that, for small $t \neq 0$, $[b_0(t) : b_1(t)]$ equals the constant $[a_0 : a_1]$, which contradicts the $j = 1$ case of Lemma 6.10. $\square$

§7. **An Example**

We will now use Proposition 6.11 and Corollary 6.8 to perform calculations in an example in which $\dim \Sigma_{\mathbf{F}^\bullet} f = 1$.

In this long example, we will look at the constant sheaf on a space which is not a local complete intersection, and consider a function with a non-isolated critical locus (in any sense of the term "critical locus"). This example is complicated enough to be interesting, and yet simple enough that we can calculate not only the LêVo modules, but also the stalk cohomology of the perverse cohomology of the vanishing cycles. Thus, we can "check" our LêVo module calculations.

In harder examples, the calculation of the LêVo modules would proceed in a similar fashion; though the intersection cycles may be significantly more difficult to compute. Nonetheless, such calculations can be carried out by computer algebra packages. While this falls short of calculating the actual stalk cohomology of the perverse cohomology of the vanishing cycles, it should be viewed as a reasonable – and effectively calculable – approximation.

Use (u, x, y, z) as coordinates for $\mathcal{U} := \mathbb{C}^4$, and let $X := V(u, x) \cup V(y, z)$. Thus, X is the simplest non-local complete intersection. The $\mathbf{F}^\bullet$ that we will use is $\mathbb{Z}^\bullet_X$.

Whenever we suppress the reference to the complex of sheaves below, it is assumed that we are using constant $\mathbb{Z}$ coefficients, and – in this case – we will write ordinary cohomology in place of hypercohomology. We continue to use (w_0, w_1, w_2, w_3) as the cotangent coordinates.

The strata of X are $S_0 := \{\mathbf{0}\}$, $S_1 := V(u, x) - \{\mathbf{0}\}$, and $S_2 := V(y, z) - \{\mathbf{0}\}$. Obviously, the dimensions of the strata are $d_0 = 0$, $d_1 = 2$, and $d_2 = 2$, respectively. The normal slices to all strata are contractible, while the corresponding complex links are given by: $\mathbb{L}_0 =$ two complex disks (sets of complex dimension one), $\mathbb{L}_1 = \emptyset$, and $\mathbb{L}_2 = \emptyset$. Thus,

$H^{k-0}(\mathbb{N}_0, \mathbb{L}_0) = 0$ unless $k = 1$, and $H^{1-0}(\mathbb{N}_0, \mathbb{L}_0) \cong \mathbb{Z}$;

$H^{k-2}(\mathbb{N}_1, \mathbb{L}_1) = 0$ unless $k = 2$, and $H^{2-2}(\mathbb{N}_1, \mathbb{L}_1) \cong \mathbb{Z}$; and

$H^{k-2}(\mathbb{N}_2, \mathbb{L}_2) = 0$ unless $k = 2$, and $H^{2-2}(\mathbb{N}_2, \mathbb{L}_2) \cong \mathbb{Z}$.

Therefore, $\mathrm{gecc}^k(\mathbb{Z}^\bullet_X)$ is zero unless $k = 1$ or $k = 2$, and we find that

$$\mathrm{gecc}^1(\mathbb{Z}^\bullet_X) = \mathbb{Z}\,[T^*_{\{\mathbf{0}\}}\mathcal{U}] = \mathbb{Z}\,[V(u, x, y, z)]\,,$$

$$\mathrm{gecc}^2(\mathbb{Z}^\bullet_X) = \mathbb{Z}\left[\overline{T^*_{S_1}\mathcal{U}}\right] + \mathbb{Z}\left[\overline{T^*_{S_2}\mathcal{U}}\right] = \mathbb{Z}\,[V(u, x, w_2, w_3)] + \mathbb{Z}\,[V(y, z, w_0, w_1)]\,.$$

Now, let $\tilde f := (u^\alpha + x^\beta)^\tau + y^\gamma + z^\delta$, where $\alpha, \beta, \gamma, \delta, \tau \geqslant 2$.

We calculate as described in Corollary 6.8. We will have to perform two separate such calculations: one where $k = 1$ and one where $k = 2$. We will follow the notation from Corollary 6.8, except that we will include a superscript on the left to indicate the value of k.

First, we find that

$$\operatorname{im} d\tilde f = V\big(w_0 - \tau\alpha(u^\alpha + x^\beta)^{\tau-1}u^{\alpha-1},\ w_1 - \tau\beta(u^\alpha + x^\beta)^{\tau-1}x^{\beta-1},\ w_2 - \gamma y^{\gamma-1},\ w_3 - \delta z^{\delta-1}\big)$$

and so

$$\operatorname{im} d\tilde f \cap |\operatorname{gecc}^1(\mathbb{Z}^\bullet_X)| = V(u, x, y, z, w_0, w_1, w_2, w_3)$$

and

$$\operatorname{im} d\tilde f \cap |\operatorname{gecc}^2(\mathbb{Z}^\bullet_X)| =$$

$$V(u, x, w_2, w_3, w_0, w_1, y, z) \cup V(y, z, w_0, w_1, u^\alpha + x^\beta, w_2, w_3) = V(u^\alpha + x^\beta, y, z, w_0, w_1, w_2, w_3).$$

As $\bigcup\limits_{v\in\mathbb{C}} \operatorname{supp}(\phi_{f-v}[-1]\mathbb{Z}^\bullet_X) = \eta(|\operatorname{gecc}^\bullet(\mathbf{F}^\bullet)| \cap \operatorname{im} d\tilde f)$, we see that $\phi_{f-v}[-1]\mathbb{Z}^\bullet_X$ is supported only where $v = 0$, and $\operatorname{supp}(\phi_f[-1]\mathbb{Z}^\bullet_X) = V(u^\alpha + x^\beta, y, z)$. Thus, $\dim_{\mathbf{0}} \operatorname{supp}(\phi_f[-1]\mathbb{Z}^\bullet_X) = 1$ and, as mentioned in Remark 6.9, we may calculate the LêVo modules as in Corollary 6.8, without worrying ahead of time about whether the coordinates are generic enough; if the intersections in the calculations are all proper, then the coordinates are generic enough.

We proceed with the calculation as described in 6.8.

Let

$${}^1\Pi^4 := \operatorname{gecc}^1(\mathbb{Z}^\bullet_X) = \mathbb{Z}\ [V(u, x, y, z)]\,.$$

Then,

$${}^1\Pi^4 \odot V\left(w_3 - \frac{\partial \tilde f}{\partial z}\right) \ = \ \mathbb{Z}\ [V(u, x, y, z)] \odot [V(w_3 - \delta z^{\delta-1})] = \mathbb{Z}\,[V(u, x, y, z, w_3)],$$

which has no components contained in $\operatorname{im} d\tilde f \cap |\operatorname{gecc}^1(\mathbb{Z}^\bullet_X)| = V(u, x, y, z, w_0, w_1, w_2, w_3)$. Thus,

$${}^1\Pi^3 = \mathbb{Z}\,[V(u, x, y, z, w_3)],$$

and we continue, to find

$${}^1\Pi^3 \odot V\left(w_2 - \frac{\partial \tilde f}{\partial y}\right) \ = \ \mathbb{Z}\,[V(u, x, y, z, w_3)] \odot [V(w_2 - \gamma y^{\gamma-1})] \ = \ \mathbb{Z}\,[V(u, x, y, z, w_3, w_2)] = {}^1\Pi^2.$$

$${}^1\Pi^2 \odot V\left(w_1 - \frac{\partial \tilde f}{\partial x}\right) \ = \ \mathbb{Z}\,[V(u, x, y, z, w_3, w_2)] \odot V(w_1 - \tau\beta(u^\alpha + x^\beta)^{\tau-1}x^{\beta-1}) =$$

$$\mathbb{Z}\,[V(u, x, y, z, w_3, w_2, w_1)] = {}^1\Pi^1.$$

$${}^1\Pi^1 \odot V\left(w_0 - \frac{\partial \tilde f}{\partial u}\right) \ = \ \mathbb{Z}\,[V(u, x, y, z, w_3, w_2, w_1)] \odot V\left(w_0 - \tau\alpha(u^\alpha + x^\beta)^{\tau-1}u^{\alpha-1}\right) =$$

$$\mathbb{Z}\left[V(u,x,y,z,w_3,w_2,w_1,w_0)\right] = {}^1\Delta^0,$$

where this last equality follows from the fact that $\operatorname{im} d\tilde{f} \cap |\operatorname{gecc}^1(\mathbb{Z}^\bullet_X)| = V(u,x,y,z,w_0,w_1,w_2,w_3)$. Note that the above calculations show that ${}^1\Delta^j = 0$ if $j \neq 0$.

The calculations for $k = 2$ are more interesting. Let

$${}^2\Pi^4 := \operatorname{gecc}^2(\mathbb{Z}^\bullet_X) = \mathbb{Z}\ [V(u,x,w_2,w_3)] + \mathbb{Z}\ [V(y,z,w_0,w_1)]\,.$$

$${}^2\Pi^4 \odot V\left(w_3 - \frac{\partial \tilde{f}}{\partial z}\right) \;=\; \Big(\mathbb{Z}\ [V(u,x,w_2,w_3)] + \mathbb{Z}\ [V(y,z,w_0,w_1)]\Big) \odot [V(w_3 - \delta z^{\delta-1})] =$$

$$(\mathbb{Z}^{\delta-1})\ [V(u,x,w_2,w_3,z)] + \mathbb{Z}\ [V(y,z,w_0,w_1,w_3)]\,.$$

As neither of these summands is contained in $\operatorname{im} d\tilde{f} \cap |\operatorname{gecc}^2(\mathbb{Z}^\bullet_X)|$, we find that

$${}^2\Pi^3 = (\mathbb{Z}^{\delta-1})\ [V(u,x,w_2,w_3,z)] + \mathbb{Z}\ [V(y,z,w_0,w_1,w_3)]$$

and we continue.

$${}^2\Pi^3 \odot V\left(w_2 - \frac{\partial \tilde{f}}{\partial y}\right) \;=\; \Big((\mathbb{Z}^{\delta-1})\ [V(u,x,w_2,w_3,z)] + \mathbb{Z}\ [V(y,z,w_0,w_1,w_3)]\Big) \odot [V(w_2 - \gamma y^{\gamma-1})] =$$

$$(\mathbb{Z}^{(\delta-1)(\gamma-1)})\ [V(u,x,w_2,w_3,z,y)] + \mathbb{Z}\ [V(y,z,w_0,w_1,w_3,w_2)] = {}^2\Pi^2.$$

$${}^2\Pi^2 \odot V\left(w_1 - \frac{\partial \tilde{f}}{\partial x}\right) =$$

$$\Big((\mathbb{Z}^{(\delta-1)(\gamma-1)})\ [V(u,x,w_2,w_3,z,y)] + \mathbb{Z}\ [V(y,z,w_0,w_1,w_3,w_2)]\Big) \odot V(w_1 - \tau\beta(u^\alpha + x^\beta)^{\tau-1}x^{\beta-1}) =$$

$$(\mathbb{Z}^{(\delta-1)(\gamma-1)})\ [V(u,x,w_2,w_3,z,y,w_1)] + \mathbb{Z}\ \left[V(y,z,w_0,w_1,w_3,w_2,(u^\alpha + x^\beta)^{\tau-1}x^{\beta-1})\right] =$$

$$(\mathbb{Z}^{(\delta-1)(\gamma-1)})\ [V(u,x,w_2,w_3,z,y,w_1)] + (\mathbb{Z}^{\beta-1})\ [V(y,z,w_0,w_1,w_3,w_2,x)] +$$

$$(\mathbb{Z}^{\tau-1})\ \left[V(y,z,w_0,w_1,w_3,w_2,u^\alpha + x^\beta)\right],$$

where this last summand is contained in $\operatorname{im} d\tilde{f}$, but the two earlier ones were not. Thus,

$${}^2\Delta^1 = (\mathbb{Z}^{\tau-1})\ \left[V(y,z,w_0,w_1,w_3,w_2,u^\alpha + x^\beta)\right]$$

and

$${}^2\Pi^1 = (\mathbb{Z}^{(\delta-1)(\gamma-1)})\ [V(u,x,w_2,w_3,z,y,w_1)] + (\mathbb{Z}^{\beta-1})\ [V(y,z,w_0,w_1,w_3,w_2,x)]\,.$$

Finally, we find

$${}^2\Pi^1 \odot V\left(w_0 - \frac{\partial \tilde{f}}{\partial u}\right) =$$

$$\Big((\mathbb{Z}^{(\delta-1)(\gamma-1)})\ [V(u,x,w_2,w_3,z,y,w_1)] + (\mathbb{Z}^{\beta-1})\ [V(y,z,w_0,w_1,w_3,w_2,x)]\Big)\odot$$

$$V(w_0 - \tau\alpha(u^\alpha + x^\beta)^{\tau-1}u^{\alpha-1}) =$$

$$(\mathbb{Z}^{(\delta-1)(\gamma-1)})\ [V(u,x,w_2,w_3,z,y,w_1,w_0)] + (\mathbb{Z}^{(\beta-1)(\alpha-1)})\ [V(y,z,w_0,w_1,w_3,w_2,x,u)] +$$
$$(\mathbb{Z}^{(\beta-1)(\tau-1)\alpha})\ [V(y,z,w_0,w_1,w_3,w_2,x,u)] =$$
$$(\mathbb{Z}^{(\delta-1)(\gamma-1)+(\beta-1)(\alpha\tau-1)})\ [V(u,x,y,z,w_0,w_1,w_2,w_3)] = {}^2\Delta^0.$$

Projecting down into $\mathcal{U}$ (and suppressing the reference to the coordinates and to $\mathbb{Z}^\bullet_X$), we find that the only non-zero LêVo cycles are:

$${}^1\Lambda^0_f = \mathbb{Z}\,[\{\mathbf{0}\}],$$

$${}^2\Lambda^1_f = (\mathbb{Z}^{\tau-1})\ \left[V(u^\alpha + x^\beta, y, z)\right],$$

and

$${}^2\Lambda^0_f = (\mathbb{Z}^{(\delta-1)(\gamma-1)+(\beta-1)(\alpha\tau-1)})\,[\{\mathbf{0}\}].$$

The corresponding LêVo modules at the origin are

$${}^1\lambda^0_f(\mathbf{0}) = \mathbb{Z}$$

$${}^2\lambda^1_f(\mathbf{0}) = \Big(V(u)\odot {}^2\Lambda^1_f\Big)_{\{\mathbf{0}\}} = \Big(V(u)\odot(\mathbb{Z}^{\tau-1})\ \left[V(u^\alpha+x^\beta,y,z)\right]\Big)_{\{\mathbf{0}\}} = \mathbb{Z}^{\beta(\tau-1)},$$

and

$${}^2\lambda^0_f(\mathbf{0}) = \mathbb{Z}^{(\delta-1)(\gamma-1)+(\beta-1)(\alpha\tau-1)}.$$

We will now calculate the stalk cohomology of the perverse cohomology of the vanishing cycles, and then compare that data with the LêVo module data. This comparison will use the Zawatsky complex (5.19) for the vanishing cycles, which was the next-to-last statement in Theorem 6.4.

For every locally closed subset $Z \subseteq X$, we let $(\mathbb{Z}^\bullet_X)_Z$ denote the restriction-extension of $\mathbb{Z}^\bullet_X$ on Z, i.e., the complex obtained by first restricting to Z and then by extending by zero over all of X.

Let $X_1 := V(u,x)$ and $X_2 := V(y,z)$. Then, there is a canonical distinguished triangle

$$\mathbb{Z}^\bullet_X \to (\mathbb{Z}^\bullet_X)_{X_1}\oplus(\mathbb{Z}^\bullet_X)_{X_2} \to (\mathbb{Z}^\bullet_X)_{\{\mathbf{0}\}} \xrightarrow{[1]} \mathbb{Z}^\bullet_X.$$

Note that $(\mathbb{Z}^\bullet_X)_{X_1}[2]$, $(\mathbb{Z}^\bullet_X)_{X_1}[2]$, and $(\mathbb{Z}^\bullet_X)_{\{\mathbf{0}\}}$ are all perverse, and so the only non-zero portion of the long exact sequence on perverse cohomology resulting from the above distinguished triangle becomes

$$0 \to {}^\mu H^0(\mathbb{Z}^\bullet_X) \to 0 \to (\mathbb{Z}^\bullet_X)_{\{\mathbf{0}\}} \to {}^\mu H^1(\mathbb{Z}^\bullet_X) \to 0 \to 0 \to {}^\mu H^2(\mathbb{Z}^\bullet_X) \to (\mathbb{Z}^\bullet_X)_{X_1}\oplus(\mathbb{Z}^\bullet_X)_{X_2} \to 0.$$

Therefore, ${}^\mu H^k(\mathbb{Z}^\bullet_X)$ is unequal to zero only when $k=1$ or $k=2$, and ${}^\mu H^1(\mathbb{Z}^\bullet_X) \cong (\mathbb{Z}^\bullet_X)_{\{\mathbf{0}\}}$ and ${}^\mu H^2(\mathbb{Z}^\bullet_X) \cong (\mathbb{Z}^\bullet_X)_{X_1}\oplus(\mathbb{Z}^\bullet_X)_{X_2}$.

It follows that

$$\phi_f[-1]\big({}^\mu H^1(\mathbb{Z}^\bullet_X)\big) \cong {}^\mu H^1(\phi_f[-1]\mathbb{Z}^\bullet_X) \cong (\mathbb{Z}^\bullet_X)_{\{\mathbf{0}\}}$$

and

$$\phi_f[-1]\big({}^\mu H^2(\mathbb{Z}^\bullet_X)\big) \cong {}^\mu H^2(\phi_f[-1]\mathbb{Z}^\bullet_X) \cong \phi_f[-1]\big((\mathbb{Z}^\bullet_X)_{X_1}\big)\oplus\phi_f[-1]\big((\mathbb{Z}^\bullet_X)_{X_2}\big).$$

If we let $f_1 := f_{|_{X_1}}$, $f_2 := f_{|_{X_2}}$, and let $j_1 : X_1 \hookrightarrow X$ and $j_2 : X_2 \hookrightarrow X$ denote the inclusions, then we can rewrite the last isomorphisms above as

$$\phi_f[-1]\big({}^\mu H^2(\mathbb{Z}^\bullet_X)\big) \cong {}^\mu H^2(\phi_f[-1]\mathbb{Z}^\bullet_X) \cong j_{1!}\big(\phi_{f_1}[-1]\mathbb{Z}^\bullet_{X_1}\big)\oplus j_{2!}\big(\phi_{f_2}[-1]\mathbb{Z}^\bullet_{X_2}\big).$$

Now, F_1, the Milnor fibre of f_1 at the origin, is the Milnor fibre of $y^\gamma + z^\delta$ in the yz-plane, and F_2, the Milnor fibre of f_2 at the origin, is the Milnor fibre of $(u^\alpha + x^\beta)^\tau$ in the ux-plane. Thus, F_1 is homotopy-equivalent to a bouquet of $(\gamma-1)(\delta-1)$ circles, and F_2 is homotopy-equivalent to the disjoint union of τ copies of a bouquet of $(\alpha-1)(\beta-1)$ circles.

We conclude from all of this that the stalk cohomology module $H^j\big({}^\mu H^k(\phi_f[-1]\mathbb{Z}^\bullet_X)\big)_{\mathbf{0}}$ is zero except for the following:

$$H^0\big({}^\mu H^1(\phi_f[-1]\mathbb{Z}^\bullet_X)\big)_{\mathbf{0}} \cong \mathbb{Z},$$

$$H^{-1}\big({}^\mu H^2(\phi_f[-1]\mathbb{Z}^\bullet_X)\big)_{\mathbf{0}} \cong \mathbb{Z}^{\tau-1},$$

and

$$H^0\big({}^\mu H^2(\phi_f[-1]\mathbb{Z}^\bullet_X)\big)_{\mathbf{0}} \cong \mathbb{Z}^{(\gamma-1)(\delta-1)+\tau(\alpha-1)(\beta-1)}.$$

Now, the Zawatsky complex, together with our earlier calculation of the LêVo modules, also tells us that only the above stalk cohomology modules could possibly be non-zero. Furthermore, the degree 1 Zawatsky complex implies that we had to have $H^0\big({}^\mu H^1(\phi_f[-1]\mathbb{Z}^\bullet_X)\big)_{\mathbf{0}} \cong {}^1\lambda^0_f(\mathbf{0}) \cong \mathbb{Z}$.

The degree 2 Zawatsky complex tells us that there is a homomorphism

$$\omega : \mathbb{Z}^{\beta(\tau-1)} \cong {}^2\lambda^1_f(\mathbf{0}) \longrightarrow {}^2\lambda^0_f(\mathbf{0}) \cong \mathbb{Z}^{(\delta-1)(\gamma-1)+(\beta-1)(\alpha\tau-1)}$$

such that $\ker\omega \cong H^{-1}\big({}^\mu H^2(\phi_f[-1]\mathbb{Z}^\bullet_X)\big)_{\mathbf{0}}$ and $\operatorname{coker}\omega \cong H^0\big({}^\mu H^2(\phi_f[-1]\mathbb{Z}^\bullet_X)\big)_{\mathbf{0}}$.

This is consistent with our perverse cohomology calculations since:

- $H^{-1}\big({}^\mu H^2(\phi_f[-1]\mathbb{Z}^\bullet_X)\big)_{\mathbf{0}}$ is free, of rank no more than $\beta(\tau-1)$;
- $H^0\big({}^\mu H^2(\phi_f[-1]\mathbb{Z}^\bullet_X)\big)_{\mathbf{0}}$ has rank no more than $(\delta-1)(\gamma-1)+(\beta-1)(\alpha\tau-1)$; and
- the alternating sums agree, i.e.,

$$\Big((\delta-1)(\gamma-1)+(\beta-1)(\alpha\tau-1)\Big) - \Big(\beta(\tau-1)\Big) \ = \ \Big((\gamma-1)(\delta-1)+\tau(\alpha-1)(\beta-1)\Big) - \Big(\tau-1\Big).$$

Finally, the reader should check that either method yields the same Euler characteristic for the reduced cohomology of the Milnor fibre of f at the origin; namely,

$$\widetilde{\chi}\big(F_{f,\mathbf{0}}\big) = -\alpha\beta\tau + \beta\tau + \alpha\tau - \gamma\delta + \gamma + \delta - 1.$$

§8. **Remarks and Future Directions**

What we have shown in this paper is that the characteristic polar modules provide an enriched form of the classical absolute polar multiplicities, and the Lê-Vogel modules provide an enriched form of the Lê numbers. Moreover, these enriched pieces of data encode a great deal of data associated to complexes of sheaves, and yet these enriched devices remain calculable in an algebraic/intersection-theoretic manner.

Originally, this paper was to be entitled "Enriched Cycles and Equisingularity". However, as the list of fundamental definitions and results grew, it became clear that most of the equisingularity results would have to go into another paper. We say "most" because we view Theorem 6.5, and even Corollary 3.5, as equisingularity results.

What we will prove in Enriched Cycles and Equisingularity is that the constancy of the Lê-Vogel modules throughout a family implies that the a_f condition holds, and that the stalk cohomology of the vanishing cycles is constant.

As fundamental tools in proving these results, we will prove an upper-semicontinuity result and general Lê-Iomdine-Vogel formulas. The notion of upper-semicontinuity makes sense for enriched cycles due to the existence of the partial ordering on enriched cycles given in Section 2. The Lê-Iomdine-Vogel formulas are a generalization of the Lê-Iomdine formulas of [**M3**], and use the general Vogel cycle results that we developed in Part I of [**M7**]; these formulas inductively reduce a number of problems to the case of isolated critical points.

In the future, we will also investigate the extent to which one can encode monodromy results, and prove monodromy theorems, via enriched cycles; we wish to capture many aspects of the monodromy on $\psi_f[-1]\mathbf{F}^\bullet$ and $\phi_f[-1]\mathbf{F}^\bullet$.

It may appear that we have not given ourselves enough structure to discuss the monodromies – for we actually defined two enriched cycles to be equal if the component modules were isomorphic. In other words, an enriched cycle is actually an equivalence class. This was necessary since there are no canonical choices for defining the Morse data to strata.

However, we can also define the monodromies as equivalence classes. If we fix one representative for an enriched cycle, the monodromy is determined by an automorphism of each component module. If we switch to a different representative of the enriched cycle, an equivalent family of automorphisms is one obtained by pulling-back the original family of automorphisms via the isomorphisms of the component modules, i.e., the new automorphisms are conjugate to the original ones.

It will be interesting to see to what extent we can recover portions of such deep results as the monodromy theorem and the decomposition theorem through enriched cycle techniques.

Finally, it would greatly enhance the algorithmic aspect of our work if we could prove Proposition 6.11 regardless of the dimension of $\Sigma_{\mathbf{F}^\bullet} f$. On the other hand, it would also be interesting to find a counterexample for higher-dimensional critical loci.

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