A New Invariant for 1-dimensional Hypersurface Singularities

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Abstract

We define and explore a new numerical invariant for hypersurfaces with 1-dimensional critical loci: the beta number. The beta number is an invariant of the ambient topological-type of the hypersurface, is non-negative, and is algebraically calculable.

The beta number being zero places strong restrictions on the vanishing cycles of the hypersurface; exactly how strong the restrictions are remains an open question.

1 Introduction

Throughout this paper, we suppose that $\mathcal{U}$ is an open neighborhood of the origin in $\mathbb{C}^{n+1}$, and that $f : \mathcal{U} \rightarrow (\mathbb{C},0)$ is a complex analytic function with a 1-dimensional critical locus at the origin, i.e., $\dim_0 \Sigma f = 1$. We use coordinates $(z_0, \ldots, z_n)$ on $\mathcal{U}$.

We assume that $L$ is a linear form which is generic enough so that $\dim_0 \Sigma (f_{|V(L)}) = 0$. For convenience, possibly after a linear change of coordinates, we may assume that $L$ is the first coordinate $z_0$, so that we have $\dim_0 \Sigma (f_{|V(z_0)}) = 0$.

We will define the beta invariant $\beta_f$ of $f$ (at the origin), and show that it has the following properties:

- $\beta_f$ is effectively, algebraically calculable.
- While the computation of $\beta_f$ uses a choice of $z_0$, $\beta_f$ is independent of $z_0$, as long as $\dim_0 \Sigma (f_{|V(z_0)}) = 0$. In fact, if $f$ is reduced, i.e., if $n \geq 2$, then $\beta_f$ is an invariant of the ambient topological-type of the hypersurface $V(f)$ at the origin.
- $\beta_f \geq 0$.
- $\beta_f = 0$ implies strong restrictions on the perverse sheaf of vanishing cycles.

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In particular, if all components of $\Sigma f$ are smooth at the origin, then $\beta_f = 0$ if and only if $\Sigma f$ has a single smooth component at the origin, along which the Milnor number of a hyperplane slice is constant.

It is an open question whether or not the requirement that all components of $\Sigma f$ are smooth at the origin is necessary to reach the conclusion of the final property above.

We thank Javier Bobadilla for asking the question which led to considering $\beta_f$, and Lê Dũng Tráng for valuable conversations on this topic.

2 Definition and Examples

Since

$$\Sigma(f_{|V(z_0)}) = V\left(z_0, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right),$$

our assumption that $\dim_0 \Sigma(f_{|V(z_0)}) = 0$ is precisely equivalent to saying that $V\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)$ is purely 1-dimensional at the origin (which includes possibly being empty) and is properly intersected at the origin by $V(z_0)$.

In terms of analytic cycles,

$$V\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right) = \Gamma^1_{f,z_0} + \Lambda^1_{f,z_0},$$

where $\Gamma^1_{f,z_0}$ is the relative polar curve of $f$, which consists of components not contained in $\Sigma f$, and $\Lambda^1_{f,z_0}$ is the 1-dimensional Lê cycle, which consists of components which are contained in $\Sigma f$. See Definition 1.11 of [8].

Note that $V\left(\frac{\partial f}{\partial z_0}\right)$ necessarily intersects $\Gamma^1_{f,z_0}$ properly at $0$, and that $V(z_0)$ intersects $\Lambda^1_{f,z_0}$ properly at $0$ by our assumption.

Letting $C$’s denote the underlying reduced components of $\Sigma f$ at $0$, at the origin, we have

$$\Lambda^1_{f,z_0} = \sum_C \mu_C[C],$$

where we use the square brackets to indicate that we are considering $C$ as a cycle, and $\mu_C$ is the Milnor number of $f$, restricted to a generic hyperplane slice, at a point $p$ on $C - \{0\}$ close to $0$. See Remark 1.19 of [8].

The intersection numbers $\left(\Gamma^1_{f,z_0} \cdot V\left(\frac{\partial f}{\partial z_0}\right)\right)_0$ and $\left(\Lambda^1_{f,z_0} \cdot V(z_0)\right)_0$ are the Lê numbers $\lambda^0_{f,z_0}$ and $\lambda^1_{f,z_0}$ (at the origin). See Definition 1.11 of [8].
Definition 2.1. We define

\[ \beta_{f,z_0} = \left( \Gamma_{f,z_0} \cdot V \left( \frac{\partial f}{\partial z_0} \right) \right)_0 - \sum_C \overset{\circ}{\mu}_C \left[ (C \cdot V(z_0))_0 - 1 \right] = \lambda_{f,z_0}^0 - \lambda_{f,z_0}^1 + \sum_C \overset{\circ}{\mu}_C. \]

Remark 2.2. We shall drop the \( z_0 \) from the notation \( \beta_{f,z_0} \) in the next section, where we show that the value is independent of \( z_0 \).

Example 2.3. Suppose that all of the components \( C \) of \( \Sigma f \) are smooth and transversely intersected by \( V(z_0) \) at \( 0 \). Then,

\[ \beta_{f,z_0} = \left( \Gamma_{f,z_0} \cdot V \left( \frac{\partial f}{\partial z_0} \right) \right)_0 = \lambda_{f,z_0}^0. \]

Thus, the only time that \( \beta_{f,z_0} \) really is a “new” invariant is when the critical locus itself has a singular component.

Example 2.4. Suppose \( f = z^2 + (y^2 - x^3)^d \), where \( d \geq 2 \). Both \( f|_{V(x)} \) and \( f|_{V(y)} \) have isolated critical points at the origin. We will calculate both \( \beta_{f,x} \) and \( \beta_{f,y} \), and see that they are the same.

First, we find that, as sets,

\[ \Sigma f = V \left( \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = V(y^2 - x^3, z). \]

Now, as cycles, we calculate

\[ V \left( \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = V(d(y^2 - x^3)^{d-1} \cdot (-3x^2), d(y^2 - x^3)^d - 1 \cdot 2y, 2z) = V(y^2 - x^3, z). \]

Thus, we have \( \Gamma_{f,x} = V(y, z) \), and that \( \Sigma f \) consists of the single component \( C = V(y^2 - x^3, z) \), with \( \overset{\circ}{\mu}_C = (d - 1) \). Therefore,

\[ \beta_{f,x} = \left( \Gamma_{f,x} \cdot V \left( \frac{\partial f}{\partial x} \right) \right)_0 - \sum_C \overset{\circ}{\mu}_C \left[ (C \cdot V(x))_0 - 1 \right] = \]
\begin{align*}
(V(y, z) \cdot V(d(y^2 - x^3)^{d-1}(-3x^2)))_0 & - (d-1)(V(y^2 - x^3, z) \cdot V(x))_0 = \\
(V(y, z) \cdot V((y^2 - x^3)^{d-1}))_0 + (V(y, z) \cdot V(x^2))_0 - 2(d-1) &= 3(d-1) + 2 - 2(d-1) = d + 1.
\end{align*}

To calculate $\beta_{f,y}$, we proceed similarly.

As cycles, we calculate
\[
V \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial z} \right) = V(d(y^2 - x^3)^{d-1}(-3x^2), 2z) = 2V(x, z) + (d-1)V(y^2 - x^3, z) = \Gamma^1_{f,y} + \Lambda^1_{f,y}.
\]

Thus, we have $\Gamma^1_{f,y} = 2V(x, z)$, and, of course, that $\Sigma f$ consists of the single component $C = V(y^2 - x^3, z)$, with $\hat{\mu}_C = (d-1)$. Therefore,
\[
\beta_{f,y} = \left( \Gamma^1_{f,y} \cdot V \left( \frac{\partial f}{\partial y} \right) \right)_0 - \sum_C \hat{\mu}_C \left[ (C \cdot V(y))_0 - 1 \right] =
\]
\[
(2V(x, z) \cdot V(d(y^2 - x^3)^{d-1}(2y)))_0 - (d-1)(V(y^2 - x^3, z) \cdot V(y))_0 =
\]
\[
(2V(x, z) \cdot V((y^2 - x^3)^{d-1}))_0 + (2V(x, z) \cdot V(y))_0 - 3(d-1) = 4(d-1) + 2 - 3(d-1) = d + 1.
\]

As promised, we see that $\beta_{f,x} = \beta_{f,y}$, even though the separate terms in the calculation are different.

## 3 Invariance

In this section, we prove the invariance of $\beta_{f,z_0}$, so long as $\dim_0 \Sigma f \setminus (V (z_0)) = 0$, and we prove the also prove the topological invariance for reduced $f$. Note that, since we are assuming that $\dim_0 \Sigma f = 1$, the only way for $f$ to be non-reduced is for $n$ to be 1, in which case $f$ would define a non-reduced plane curve.
Theorem 3.1. Suppose that $z_0$ and $\hat{z}_0$ are two linear forms such that $\dim \Sigma(f_{\mid V(z_0)}) = 0$ and $\dim \Sigma(f_{\mid V(\hat{z}_0)}) = 0$. Then, $\beta_{f,z_0} = \beta_{f,\hat{z}_0}$, and consequently we denote this common value simply by $\beta_f$.

Furthermore, if $f : (U,0) \to (\mathbb{C},0)$ and $g : (U,0) \to (\mathbb{C},0)$ are reduced with 1-dimensional critical loci at the origin, and $V(f)$ and $V(g)$ have the same local ambient topological-type at 0, then $\beta_f = \beta_g$.

Proof. Let $F_{f,0}$ denote the Milnor fiber of $f$ at 0, and let $\tilde{b}_i(F_{f,0})$ denote its $i$-th reduced Betti number.

Assume that $\dim \Sigma(f_{\mid V(z_0)}) = 0$. Then, as we showed in Theorem 3.3 of [8],
\[
\lambda^0_{f,z_0} - \lambda^1_{f,z_0} = \tilde{b}_n(F_{f,0}) - \tilde{b}_{n-1}(F_{f,0})
\]
and, hence,
\[
\beta_{f,z_0} = \lambda^0_{f,z_0} - \lambda^1_{f,z_0} + \sum_C \tilde{\mu}_C = \tilde{b}_n(F_{f,0}) - \tilde{b}_{n-1}(F_{f,0}) + \sum_C \tilde{\mu}_C.
\]
This last expression is clearly independent of the choice of $z_0$, which proves the first part of theorem.

As Lê proved in [4] and [5], the homotopy-type of the Milnor fiber is an invariant of the ambient topological-type for reduced functions $f$; thus, the topological invariance of $\beta_f$ would follow from the topological invariance of $\sum_C \tilde{\mu}_C$. However, this latter topological invariance is easy to establish.

The singular set of $V(f)$ must map to the singular set under an ambient homeomorphism and, as we require the origin to map to the origin, the punctured singular set $\Sigma V(f) - \{0\}$ must map to the punctured singular set, and so the components of $\Sigma f$ at the origin must map bijectively to the components of the singular set at the origin. Now the homotopy-type of the Milnor fiber of $f$ at a point $p \in \Sigma V(f)$ near 0 is invariant under an ambient homeomorphism, and this homotopy-type is that of a bouquet of $\tilde{\mu}_C$ $(n-1)$-spheres, where $C$ is the component of $\Sigma V(f)$ containing $p$. □

4 Non-negativity

In this section, our sole result is:

Theorem 4.1. $\beta_f \geq 0$, with equality holding if and only if $\tilde{H}^n(F_{f,0}) = 0$ and $\tilde{H}^{n-1}(F_{f,0}) \cong \mathbb{Z}^\sigma$, where $\sigma = \sum_C \tilde{\mu}_C$ and we are using cohomology with integral coefficients.
Proof. As we showed in Corollary 10.10 of [8] (though there is an indexing typographical error), there is a map \( \mathbb{Z}^{\lambda_{j,z_0}^1} \to \mathbb{Z}^{\lambda_{j,z_0}^0} \) such that \( \ker \delta \cong \tilde{H}^{n-1}(F_{f,0}) \) and \( \coker \delta \cong \tilde{H}^n(F_{f,0}) \). Thus, as

\[
\lambda_{j,z_0}^1 = \sum_C \mu_C (C \cdot V(z_0))_0,
\]

the inequality would follow from

\[(†) \quad \text{rank} \tilde{H}^{n-1}(F_{f,0}) \leq \sum_C \mu_C, \]

for then we would have

\[
\lambda_{j,z_0}^0 \geq \text{rank}(\text{im } \delta) = \lambda_{j,z_0}^1 - \text{rank}(\ker \delta) \geq \sum_C \mu_C (C \cdot V(z_0))_0 - \sum_C \mu_C.
\]

But (†) is a result of Siersma in [10], or an easy exercise using perverse sheaves (see the remark at the end of [10]). This proves the inequality.

We want to establish the claim about \( \beta_f = 0 \).

The argument above tells us that \( \beta_f = 0 \) if and only if

\[
\text{rank} \tilde{H}^{n-1}(F_{f,0}) = \sum_C \mu_C \quad \text{and} \quad \lambda_{j,z_0}^0 = \text{rank}(\text{im } \delta).
\]

This last equality holds if and only if \( \tilde{H}^n(F_{f,0}) \cong \coker \delta \) consists purely of torsion. We need to eliminate the possibility of torsion in \( \tilde{H}^n(F_{f,0}) \), i.e., the possibility of torsion in \( \tilde{H}_{n-1}(F_{f,0}) \). To do this, we discuss some of the above results in the case where the base ring is the finite field \( \mathbb{Z}/p \) for a prime \( p \).

As we showed in [9], even with coefficients in \( \mathbb{Z}/p \), there is still a map

\[
(\mathbb{Z}/p)^{\lambda_{j,z_0}^1} \to (\mathbb{Z}/p)^{\lambda_{j,z_0}^0}
\]

such that \( \ker \delta^p \cong \tilde{H}^{n-1}(F_{f,0}; \mathbb{Z}/p) \) and \( \coker \delta^p \cong \tilde{H}^n(F_{f,0}; \mathbb{Z}/p) \). Furthermore, the perverse sheaf argument which is discussed at the end of [10] is independent of the base ring, and so we conclude

\[(†). \quad \text{rank} \tilde{H}^{n-1}(F_{f,0}; \mathbb{Z}/p) \leq \sum_C \mu_C, \]

Consequently, \( \beta_f = 0 \) implies that \( \dim \tilde{H}^n(F_{f,0}; \mathbb{Z}/p) = 0 \), i.e., \( \tilde{H}^n(F_{f,0}) \) has no \( p \)-torsion. \( \square \)
Remark 4.2. With a bit more work, one can show that $\beta_f = 0$ if and only if, in the Abelian category of perverse sheaves, the vanishing cycles along $f$, restricted to $\Sigma f$, is isomorphic to the direct sum of the constant sheaves $(\mathbb{Z}_C^\bullet[1])^{\mu C}$.

As we saw in Example 2.3, if all of the components of $\Sigma f$ are smooth at $0$, then, for generic $z_0$, $\beta_f = \lambda_{f,z_0}^0$; then, results on $\lambda_{f,z_0}^0$ imply strong results when $\beta_f = 0$ or 1.

For instance, the non-splitting result of Lê in [6], Lazzeri in [3], or Gabrielov in [2] immediately implies the first item below, while the main theorem of [7] immediately implies the second item.

Proposition 4.3. Suppose that all of the components of $\Sigma f$ are smooth at $0$.

1. If $\beta_f = 0$, then $\Sigma f$ has a unique component at the origin, along which the Milnor number of a generic hyperplane slice is constant.

2. If $\beta_f = 1$, then $\tilde{H}^n(F_{f,0}) = 0$ and rank $\tilde{H}^{n-1}(F_{f,0}) \cong \mathbb{Z}_{\sigma-1}$, where $\sigma = \sum_C \hat{\mu}_C$.

5 The Open Question

As we stated earlier, the real interest in the beta invariant is when the critical locus of $f$ is itself singular. In fact, the reason we defined the beta invariant is because it arose naturally while we were considering a question posed by J. Bobadilla:

Question 5.1. Suppose that the critical locus of $f$ has a single 1-dimensional irreducible component $C$ at the origin, along which the vanishing cycles of $f$ are constant (i.e., $\beta_f = 0$). Is it true that $C$ must be smooth?

This question is related to Lê’s conjecture (see, for instance, [1]), and the suspicion (not yet a conjecture) is that the answer to this question is “yes”. Furthermore, the suspicion is also that the proof of this will be very difficult, and will require new techniques.

Of course, one can generalize the question to the case where the critical locus is not assumed to be irreducible:

Question 5.2. Suppose that the critical locus of $f$ is 1-dimensional at the origin, and $\beta_f = 0$. Is it true that $\Sigma f$ must be smooth (and, hence, irreducible)?
It is conceivable that the answer to this more general question is “no”, and yet the answer to the original question is “yes”. Good candidates for counterexamples are hard to produce.
References


