CRITICAL POINTS OF FUNCTIONS ON SINGULAR SPACES

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Abstract. We compare and contrast various notions of the "critical locus" of a complex analytic function on a singular space. After choosing a topological variant as our primary notion of the critical locus, we justify our choice by generalizing Lê and Saito's result that constant Milnor number implies that Thom's a_f condition is satisfied.

§0. Introduction.

Let $\mathcal{U}$ be an open subset of $\mathbb{C}^{n+1}$, let $\mathbf{z} := (z_0, z_1, \dots, z_n)$ be coordinates for $\mathbb{C}^{n+1}$, and suppose that $\tilde{f} : \mathcal{U} \to \mathbb{C}$ is an analytic function. Then, all conceivable definitions of the *critical locus*, $\Sigma\tilde{f}$, of $\tilde{f}$ agree: one can consider the points, $\mathbf{x}$, where the derivative vanishes, i.e., $d_{\mathbf{x}}\tilde{f} = 0$, or one can consider the points, $\mathbf{x}$, where the Taylor series of $\tilde{f}$ at $\mathbf{x}$ has no linear term, i.e., $\tilde{f} - \tilde{f}(\mathbf{x}) \in \mathfrak{m}^2_{\mathcal{U},\mathbf{x}}$ (where $\mathfrak{m}_{\mathcal{U},\mathbf{x}}$ is the maximal ideal in the coordinate ring of $\mathcal{U}$ at $\mathbf{x}$), or one can consider the points, $\mathbf{x}$, where the Milnor fibre of $\tilde{f}$ at $\mathbf{x}$, $F_{\tilde{f},\mathbf{x}}$, is not trivial (where, here, "trivial" could mean even up to analytic isomorphism).

Now, suppose that X is an analytic subset of $\mathcal{U}$, and let $f := \tilde{f}_{|_X}$. Then, what should be meant by "the critical locus of f"? It is not clear what the relationship is between points, $\mathbf{x}$, where $f - f(\mathbf{x}) \in \mathfrak{m}^2_{X,\mathbf{x}}$ and points where the Milnor fibre, $F_{f,\mathbf{x}}$, is not trivial (with any definition of trivial); moreover, the derivative $d_{\mathbf{x}}f$ does not even exist.

We are guided by the successes of Morse Theory and stratified Morse Theory to choosing the Milnor fibre definition as our primary notion of critical locus, for we believe that critical points should coincide with changes in the topology of the level hypersurfaces of f. Therefore, we make the following definition:

Definition 0.1. The $\mathbb{C}$-*critical locus of* f, $\Sigma_{\mathbb{C}}f$, is given by

$$\Sigma_{\mathbb{C}}f := \{\mathbf{x} \in X \mid H^*(F_{f,\mathbf{x}};\ \mathbb{C}) \neq H^*(point;\ \mathbb{C})\}.$$

(The reasons for using field coefficients, rather than $\mathbb{Z}$, are technical: we want Lemma 3.1 to be true.)

In Section 1, we will compare and contrast the $\mathbb{C}$-critical locus with other possible notions of critical locus, including the ones mentioned above and the stratified critical locus.

After Section 1, the remainder of this paper is dedicated to showing that Definition 0.1 really yields a useful, calculable definition of the critical locus. We show this by looking at the case of a generalized isolated singularity, i.e., an isolated point of $\Sigma_{\mathbb{C}}f$, and showing that, at such a point, there is a workable definition of the Milnor number(s) of f; we show that the Betti numbers of the Milnor fibre can be calculated (3.7.ii), and we give a generalization of the result of Lê and Saito [**L-S**] that constant Milnor

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number throughout a family implies Thom's a_f condition holds. Specifically, in Corollary 5.14, we prove (with slightly weaker hypotheses) that:

Theorem 0.2. *Let W be a (not necessarily purely) d-dimensional analytic subset of an open subset of $\mathbb{C}^n$. Let Z be a d-dimensional irreducible component of W. Let $X := \overset{\circ}{\mathbb{D}} \times W$ be the product of an open disk about the origin with W, and let $Y := \overset{\circ}{\mathbb{D}} \times Z$.*

Let $f : (X, \overset{\circ}{\mathbb{D}} \times \{\mathbf{0}\}) \to (\mathbb{C}, 0)$ be an analytic function, and let $f_t(\mathbf{z}) := f(t, \mathbf{z})$. Suppose that f_0 is in the square of the maximal ideal of Z at $\mathbf{0}$.

Suppose that $\mathbf{0}$ is an isolated point of $\Sigma_{\mathbb{C}}(f_0)$, and that the reduced Betti number $\tilde{b}_{d-1}(F_{f_a,(a,\mathbf{0})})$ is independent of a for all small a.

Then, $\tilde{b}_{d-1}(F_{f_a,(a,\mathbf{0})}) \neq 0$ and, near $\mathbf{0}$, $\Sigma(f_{|_{Y_{\mathrm{reg}}}}) \subseteq \overset{\circ}{\mathbb{D}} \times \{\mathbf{0}\}$ and the pair $(Y_{\mathrm{reg}} - \Sigma(f_{|_{Y_{\mathrm{reg}}}}),\ \overset{\circ}{\mathbb{D}} \times \{\mathbf{0}\})$ satisfies Thom's a_f condition at $\mathbf{0}$.

Thom's a_f is important for several reasons, but perhaps the best reason is because it is an hypothesis of Thom's Second Isotopy Lemma. General results on the a_f condition have been proved by many researchers: Hironaka, Lê, Saito, Henry, Merle, Sabbah, Briançon, Maisonobe, Parusiński, etc., and the above theorem is closely related to the recent results contained in [**BMM**] and [**P**]. However, the reader should contrast the hypotheses of Theorem 0.2 with those of the main theorem of [**BMM**] (Theorem 4.2.1); our main hypothesis is that a single number is constant throughout the family, while the main hypothesis of Theorem 4.2.1 of [**BMM**] is a condition which requires one to check an infinite amount of data: the property of local stratified triviality. Moreover, the Betti numbers that we require to be constant are actually calculable.

While much of this part is fairly technical in nature, there are three new, key ideas that guide us throughout.

The first of these fundamental precepts is: controlling the **vanishing cycles** in a family of functions is enough to control Thom's a_f condition and, perhaps, the topology throughout the family. While this may seem like an obvious principle – given the results of Lê and Saito in [**L-S**] and of Lê and Ramanujam in [**L-R**] – in fact, in the general setting, most of the known results seem to require the constancy of much stronger data, e.g., the constancy of the polar multiplicities [**Te**] or that one has the local stratified triviality property [**BMM**]. In a very precise sense, controlling the polar multiplicities corresponds to controlling the **nearby cycles** of the family of functions, instead of merely controlling the vanishing cycles. As we show in Corollary 4.4, controlling the characteristic cycle of the vanishing cycles is sufficient for obtaining the a_f condition.

Our second fundamental idea is: the correct setting for all of our cohomological results is where perverse sheaves are used as coefficients. While papers on intersection cohomology abound, and while perverse sheaves are occasionally used as a tool (e.g., [**BMM**, 4.2.1]), we are not aware of any other work on general singularities in which arbitrary perverse sheaves of coefficients are used in an integral fashion throughout. The importance of perverse sheaves in this paper begins with Theorem 3.2, where we give a description of the critical locus of a function with respect to a perverse sheaf.

The third new feature of this paper is the recurrent use of the perverse cohomology of a complex of sheaves. This device allows us to take our general results about perverse sheaves and translate them into statements about the constant sheaf. The reason that we use perverse cohomology, instead of intersection cohomology, is because perverse cohomology has such nice functorial properties: it commutes with Verdier dualizing, and with taking nearby and vanishing cycles (shifted by $[-1]$). If we were only interested in proving results for local complete intersections (l.c.i.'s), we would never need the perverse cohomology; however, we want to prove completely general results. The perverse cohomology seems to be a hitherto unused tool for accomplishing this goal.

This paper is organized as follows:

In Section 1, we discuss seven different notions of the "critical locus" of a function. We give examples to show that, in general, all of these notions are different.

Section 2 is devoted to proving an "index theorem", Theorem 2.10, which provides the main link between the topological data of the Milnor fibre and the algebraic data obtained by blowing-up the image of $d\tilde{f}$ inside the appropriate space. This theorem is presented with coefficients in a bounded, constructible complex of sheaves; this level of generality is absolutely necessary in order to obtain the results in the remainder of this paper.

Section 3 uses the index theorem of Section 2 to show that $\Sigma_{\mathbb{C}} f$ and the Betti numbers of the Milnor fibre really are fairly well-behaved. This is accomplished by applying Theorem 2.10 in the case where the complex of sheaves is taken to be the perverse cohomology of the shifted constant sheaf. Perverse cohomology essentially gives us the "closest" perverse sheaf to the constant sheaf. Many of the results of Section 3 are stated for arbitrary perverse sheaves, for this seems to be the most natural setting.

Section 4 contains the necessary results from conormal geometry that we will need in order to conclude that topological data implies that Thom's a_f condition holds. The primary result of this section is Corollary 4.4, which once again relies on the index theorem from Section 2.

Section 5 begins with a discussion of "continuous families of constructible complexes of sheaves". We then prove in Theorem 5.7 that additivity of Milnor numbers occurs in continuous families of perverse sheaves, and we use this to conclude additivity of the Betti numbers of the Milnor fibres, by once again resorting to the perverse cohomology of the shifted constant sheaf. Finally, in Corollaries 5.11 and 5.12, we prove that the constancy of the Milnor/Betti number(s) throughout a family implies that the a_f condition holds – we prove this first in the setting of arbitrary perverse sheaves, and then for perverse cohomology of the shifted constant sheaf. By translating our hypotheses from the language of the derived category back into more down-to-Earth terms, we obtain Corollary 5.12, which leads to Theorem 0.2 above.

Section 1. CRITICAL AVATARS.

We continue with $\mathcal{U}$, $\mathbf{z}$, $\tilde{f}$, X, and f as in the introduction.

In this section, we will investigate seven possible notions of the "critical locus" of a function on a singular space, one of which is the $\mathbb{C}$-critical locus already defined in 0.1.

Definition 1.1. The *algebraic critical locus of* f, $\Sigma_{\text{alg}} f$, is defined by

$$\Sigma_{\text{alg}} f := \{\mathbf{x} \in X \mid f - f(\mathbf{x}) \in \mathfrak{m}^2_{X,\mathbf{x}}\}.$$

Remark 1.2. It is a trivial exercise to verify that

$$\Sigma_{\text{alg}} f = \{\mathbf{x} \in X \mid \text{there exists a local extension, } \hat{f}, \text{ of } f \text{ to } \mathcal{U} \text{ such that } d_{\mathbf{x}}\hat{f} = 0\}.$$

Note that $\mathbf{x}$ being in $\Sigma_{\text{alg}} f$ does **not** imply that **every** local extension of f has zero for its derivative at $\mathbf{x}$.

One might expect that $\Sigma_{\text{alg}} f$ is always a closed set; in fact, it need not be. Consider the example where $X := V(xy) \subseteq \mathbb{C}^2$, and $f = y_{|_X}$. We leave it as an exercise for the reader to verify that $\Sigma_{\text{alg}} f = V(y) - \{\mathbf{0}\}$.

There are five more variants of the critical locus of f that we will consider. We let X_{reg} denote the regular (or smooth) part of X and, if M is an analytic submanifold of $\mathcal{U}$, we let $T^*_M\mathcal{U}$ denote the conormal space to M in $\mathcal{U}$ (that is, the elements $(\mathbf{x},\eta)$ of the cotangent space to $\mathcal{U}$ such that $\mathbf{x}\in M$ and η annihilates the tangent space to M at $\mathbf{x}$). We let $N(X)$ denote the Nash modification of X, so that the fibre $N_{\mathbf{x}}(X)$ at $\mathbf{x}$ consists of limits of tangent planes from the regular part of X.

We also remind the reader that complex analytic spaces possess canonical Whitney stratifications (see [**Te**]).

Definition 1.3. We define the *regular critical locus of* f, $\Sigma_{\mathrm{reg}}f$, to be the critical locus of the restriction of f to X_{reg}, i.e., $\Sigma_{\mathrm{reg}}f=\Sigma\big(f_{|_{X_{\mathrm{reg}}}}\big)$.

We define the *Nash critical locus of* f, $\Sigma_{\mathrm{Nash}}f$, to be

$$\big\{\mathbf{x}\in X \mid \text{there exists a local extension, } \hat f, \text{ of } f \text{ to } \mathcal{U} \text{ such that } d_{\mathbf{x}}\hat f(T)\equiv 0, \text{ for all } T\in N_{\mathbf{x}}(X)\big\}.$$

We define the *conormal-regular critical locus of* f, $\Sigma_{\mathrm{cnr}}f$, to be

$$\big\{\mathbf{x}\in X \mid \text{there exists a local extension, } \hat f, \text{ of } f \text{ to } \mathcal{U} \text{ such that } (\mathbf{x}, d_{\mathbf{x}}\hat f)\in \overline{T^*_{X_{\mathrm{reg}}}\mathcal{U}}\ \big\};$$

it is trivial to see that this set is equal to

$$\big\{\mathbf{x}\in X \mid \text{there exists a local extension, } \hat f, \text{ of } f \text{ to } \mathcal{U} \text{ such that } d_{\mathbf{x}}\hat f(T)\equiv 0, \text{ for some } T\in N_{\mathbf{x}}(X)\big\}.$$

Let $\mathcal{S}=\{S_\alpha\}$ be a (complex analytic) Whitney stratification of X. We define the *$\mathcal{S}$-stratified critical locus of* f, $\Sigma_{\mathcal{S}}f$, to be $\bigcup_\alpha \Sigma\big(f_{|_{S_\alpha}}\big)$. If $\mathcal{S}$ is clear, we simply call $\Sigma_{\mathcal{S}}f$ the stratified critical locus.

If $\mathcal{S}$ is, in fact, the canonical Whitney stratification of X, then we write $\Sigma_{\mathrm{can}}f$ in place of $\Sigma_{\mathcal{S}}f$, and call it the *canonical stratified critical locus.*

We define the *relative differential critical locus of* f, $\Sigma_{\mathrm{rdf}}f$, to be the union of the singular set of X and $\Sigma_{\mathrm{reg}}f$.

If $\mathbf{x}\in X$ and $h_1,\dots,h_j$ are equations whose zero-locus defines X near $\mathbf{x}$, then $\mathbf{x}\in\Sigma_{\mathrm{rdf}}f$ if and only if the rank of the Jacobian map of $(\tilde f, h_1,\dots,h_j)$ at $\mathbf{x}$ is not maximal among all points of X near $\mathbf{x}$. By using this Jacobian, we could (but will not) endow $\Sigma_{\mathrm{rdf}}f$ with a scheme structure (the *critical space*) which is independent of the choice of the extension $\tilde f$ and the defining functions $h_1,\dots,h_n$ (see [**Lo**, 4.A]). The proof of the independence uses relative differentials; this is the reason for our terminology.

Remark 1.4. In terms of conormal geometry, $\Sigma_{\mathcal{S}}f=\Big\{\mathbf{x}\in X \mid (\mathbf{x}, d_{\mathbf{x}}\tilde f)\in\bigcup_\alpha T^*_{S_\alpha}\mathcal{U}\Big\}$ or, using Whitney's condition a) again, $\Sigma_{\mathcal{S}}f=\Big\{\mathbf{x}\in X \mid (\mathbf{x}, d_{\mathbf{x}}\tilde f)\in\bigcup_\alpha \overline{T^*_{S_\alpha}\mathcal{U}}\Big\}$.

Clearly, $\Sigma_{\mathrm{rdf}}f$ is closed, and it is an easy exercise to show that Whitney's condition a) implies that $\Sigma_{\mathcal{S}}f$ is closed. On the other hand, $\Sigma_{\mathrm{reg}}f$ is, in general, not closed and, in order to have any information at singular points of X, we will normally look at its closure $\overline{\Sigma_{\mathrm{reg}}f}$.

Looking at the definition of $\Sigma_{\mathrm{cnr}}f$, one might expect that $\overline{\Sigma_{\mathrm{reg}}f}=\Sigma_{\mathrm{cnr}}f$. In fact, we shall see in Example 1.8 that this is false. That $\Sigma_{\mathrm{cnr}}f$ is, itself, closed is part of the following proposition. (Recall that $\tilde f$ is our fixed extension of f to all of $\mathcal{U}$.)

In the following proposition, we show that, in the definitions of the Nash and conormal-regular critical loci, we could have used "for all" in place of "there exists" for the local extensions; in particular, this implies that we can use the fixed extension $\tilde{f}$. Finally, we show that the conormal-regular critical locus is closed.

Proposition 1.5. *The Nash critical locus of f is equal to*

$$\big\{\mathbf{x}\in X \mid \textit{for all local extensions, } \hat{f}, \textit{ of } f \textit{ to } \mathcal{U},\ d_{\mathbf{x}}\hat{f}(T)\equiv 0, \textit{ for all } T\in N_{\mathbf{x}}(X)\big\} =$$

$$\big\{\mathbf{x}\in X \mid d_{\mathbf{x}}\tilde{f}(T)\equiv 0,\ \textit{for all } T\in N_{\mathbf{x}}(X)\big\}.$$

The conormal-regular critical locus of f is equal to

$$\big\{\mathbf{x}\in X \mid \textit{for all local extensions, } \hat{f}, \textit{ of } f \textit{ to } \mathcal{U},\ (\mathbf{x}, d_{\mathbf{x}}\hat{f})\in \overline{T^*_{X_{\mathrm{reg}}}\mathcal{U}}\ \big\}$$

$$=\big\{\mathbf{x}\in X \mid (\mathbf{x}, d_{\mathbf{x}}\tilde{f})\in \overline{T^*_{X_{\mathrm{reg}}}\mathcal{U}}\big\}.$$

In addition, $\Sigma_{\mathrm{cnr}}f$ is closed.

Proof. Let $Z := \big\{\mathbf{x}\in X \mid \text{for all local extensions, } \hat{f}, \text{ of } f \text{ to } \mathcal{U},\ d_{\mathbf{x}}\hat{f}(T)\equiv 0, \text{ for all } T\in N_{\mathbf{x}}(X)\big\}$. Clearly, we have $Z\subseteq \Sigma_{\mathrm{Nash}}f$.

Suppose now that $\mathbf{x}\in\Sigma_{\mathrm{Nash}}f$. Then, there exists a local extension, $\hat{f}$, of f to $\mathcal{U}$ such that $d_{\mathbf{x}}\hat{f}(T)\equiv 0$, for all $T\in N_{\mathbf{x}}(X)$. Let $\check{f}$ be another local extension of f to $\mathcal{U}$ and let $T_\infty\in N_{\mathbf{x}}(X)$; to show that $\mathbf{x}\in Z$, what we must show is that $d_{\mathbf{x}}\check{f}(T_\infty)\equiv 0$.

Suppose not. Then, there exists $\mathbf{v}\in T_\infty$ such that $d_{\mathbf{x}}\check{f}(\mathbf{v})\neq 0$, but $d_{\mathbf{x}}\hat{f}(\mathbf{v})=0$. Therefore, there exist $\mathbf{x}_i\in X_{\mathrm{reg}}$ and $\mathbf{v}_i\in T_{\mathbf{x}_i}X_{\mathrm{reg}}$ such that $\mathbf{x}_i\to\mathbf{x}$, $T_{\mathbf{x}_i}X_{\mathrm{reg}}\to T_\infty$, and $\mathbf{v}_i\to\mathbf{v}$.

Let $\mathcal{V}$ be an open neighborhood of x in $\mathcal{U}$ which in $\hat{f}$ and $\check{f}$ are both defined. Let $\Phi:\mathcal{V}\cap\overline{TX_{\mathrm{reg}}}\to\mathbb{C}$ be defined by $\Phi(\mathbf{p},\mathbf{w})=d_{\mathbf{p}}(\hat{f}-\check{f})(\mathbf{w})$. Then, Φ is continuous, and so $\Phi^{-1}(0)$ is closed. As $(\hat{f}-\check{f})_{|X}\equiv 0$, $(\mathbf{x}_i,\mathbf{v}_i)\in\Phi^{-1}(0)$, and thus $(\mathbf{x},\mathbf{v})\in\Phi^{-1}(0)$ – a contradiction. Therefore, $Z=\Sigma_{\mathrm{Nash}}f$.

It follows immediately that $\Sigma_{\mathrm{Nash}}f=\big\{\mathbf{x}\in X \mid d_{\mathbf{x}}\tilde{f}(T)\equiv 0,\ \text{for all } T\in N_{\mathbf{x}}(X)\big\}$.

Now, let $W := \big\{\mathbf{x}\in X \mid \text{for all local extensions, } \hat{f}, \text{ of } f \text{ to } \mathcal{U},\ (\mathbf{x}, d_{\mathbf{x}}\hat{f})\in\overline{T^*_{X_{\mathrm{reg}}}\mathcal{U}}\ \big\}$. Clearly, we have $W\subseteq\Sigma_{\mathrm{cnr}}f$.

Suppose now that $\mathbf{x}\in\Sigma_{\mathrm{cnr}}f$. Then, there exists a local extension, $\hat{f}$, of f to $\mathcal{U}$ such that $(\mathbf{x}, d_{\mathbf{x}}\hat{f})\in\overline{T^*_{X_{\mathrm{reg}}}\mathcal{U}}$. Let $(\mathbf{x}_i,\eta_i)\in T^*_{X_{\mathrm{reg}}}\mathcal{U}$ be such that $(\mathbf{x}_i,\eta_i)\to(\mathbf{x}, d_{\mathbf{x}}\hat{f})$. Let $\check{f}$ be another local extension of f to $\mathcal{U}$; to show that $\mathbf{x}\in W$, what we must show is that $(\mathbf{x}, d_{\mathbf{x}}\check{f})\in\overline{T^*_{X_{\mathrm{reg}}}\mathcal{U}}$.

Since $(\check{f}-\hat{f})_{|X}\equiv 0$, for all $\mathbf{q}\in X_{\mathrm{reg}}$, $\big(\mathbf{q}, d_{\mathbf{q}}(\check{f}-\hat{f})\big)\in T^*_{X_{\mathrm{reg}}}\mathcal{U}$; in particular, $\big(\mathbf{x}_i, d_{\mathbf{x}_i}(\check{f}-\hat{f})\big)\in T^*_{X_{\mathrm{reg}}}\mathcal{U}$. Thus, $\big(\mathbf{x}_i,\eta_i+d_{\mathbf{x}_i}(\check{f}-\hat{f})\big)\in T^*_{X_{\mathrm{reg}}}\mathcal{U}$, and $\big(\mathbf{x}_i,\eta_i+d_{\mathbf{x}_i}(\check{f}-\hat{f})\big)\to(\mathbf{x}, d_{\mathbf{x}}\check{f})$. Therefore, $(\mathbf{x}, d_{\mathbf{x}}\check{f})\in\overline{T^*_{X_{\mathrm{reg}}}\mathcal{U}}$, and $W=\Sigma_{\mathrm{cnr}}f$.

It follows immediately that $\Sigma_{\mathrm{cnr}}f=\big\{\mathbf{x}\in X \mid (\mathbf{x}, d_{\mathbf{x}}\tilde{f})\in\overline{T^*_{X_{\mathrm{reg}}}\mathcal{U}}\big\}$.

Finally, we need to show that $\Sigma_{\mathrm{cnr}}f$ is closed. Let $\Psi: X\to T^*\mathcal{U}$ be given by $\Psi(\mathbf{x})=(\mathbf{x}, d_{\mathbf{x}}\tilde{f})$. Then, Ψ is a continuous map and, by the above, $\Sigma_{\mathrm{cnr}}f=\Psi^{-1}(\overline{T^*_{X_{\mathrm{reg}}}\mathcal{U}})$. □

Proposition 1.6. *There are inclusions*

$$\overline{\Sigma_{\mathrm{reg}} f} \ \subseteq \ \overline{\Sigma_{\mathrm{alg}} f} \ \subseteq \ \overline{\Sigma_{\mathrm{Nash}} f} \ \subseteq \ \Sigma_{\mathrm{cnr}} f \ \subseteq \ \overline{\Sigma_{\mathbb{C}} f} \ \subseteq \ \Sigma_{\mathrm{can}} f \ \subseteq \ \Sigma_{\mathrm{rdf}} f.$$

In addition, if $\mathcal{S}$ is a Whitney stratification of X, then $\Sigma_{\mathrm{can}} f \ \subseteq \ \Sigma_{\mathcal{S}} f$.

Proof. Clearly, $\Sigma_{\mathrm{reg}} f \ \subseteq \ \Sigma_{\mathrm{alg}} f \ \subseteq \ \Sigma_{\mathrm{Nash}} f \ \subseteq \ \Sigma_{\mathrm{cnr}} f$, and so the containments for their closures follows (recall, also that $\Sigma_{\mathrm{cnr}} f$ is closed). It is also obvious that $\Sigma_{\mathrm{can}} f \ \subseteq \ \Sigma_{\mathrm{rdf}} f$ and $\Sigma_{\mathrm{can}} f \ \subseteq \ \Sigma_{\mathcal{S}} f$.

That $\Sigma_{\mathbb{C}} f \ \subseteq \ \Sigma_{\mathrm{can}} f$ follows from Stratified Morse Theory [**Go-Mac1**], and so, since $\Sigma_{\mathrm{can}} f$ is closed, $\overline{\Sigma_{\mathbb{C}} f} \ \subseteq \ \Sigma_{\mathrm{can}} f$.

It remains for us to show that $\Sigma_{\mathrm{cnr}} f \ \subseteq \ \overline{\Sigma_{\mathbb{C}} f}$. Unfortunately, to reach this conclusion, we must refer ahead to Theorem 3.6, from which it follows immediately. (However, that $\overline{\Sigma_{\mathrm{alg}} f} \ \subseteq \ \overline{\Sigma_{\mathbb{C}} f}$ follows from A'Campo's Theorem [**A'C**].) □

Remark 1.7. For a fixed stratification $\mathcal{S}$, for all $\mathbf{x} \in X$, there exists a neighborhood $\mathcal{W}$ of $\mathbf{x}$ in X such that $\mathcal{W} \cap \Sigma_{\mathcal{S}} f \subseteq V(f - f(\mathbf{x}))$. This is easy to show: the level hypersurfaces of f close to $V(f - f(\mathbf{x}))$ will be transverse to all of the strata of $\mathcal{S}$ near $\mathbf{x}$. All of our other critical loci which are contained in $\Sigma_{\mathcal{S}} f$ (i.e., all of them except $\Sigma_{\mathrm{rdf}} f$) also satisfy this local isolated critical value property.

Example 1.8. In this example, we wish to look at the containments given in Proposition 1.6, and investigate whether the containments are proper, and also investigate what would happen if we did not take closures in the four cases where we do.

The same example that we used in Remark 1.2 shows that none of $\Sigma_{\mathrm{reg}} f$, $\Sigma_{\mathrm{alg}} f$, $\Sigma_{\mathrm{Nash}} f$, or $\Sigma_{\mathbb{C}} f$ are necessarily closed; if $X := V(xy) \subseteq \mathbb{C}^2$, and $f = y_{|_X}$, then all four critical sets are precisely $V(y) - \{\mathbf{0}\}$. Additionly, since $\Sigma_{\mathrm{cnr}} f = V(y)$, this example also shows that, in general, $\Sigma_{\mathrm{cnr}} f \ \not\subseteq \ \Sigma_{\mathbb{C}} f$.

If we continue with $X = V(xy)$ and let $g := (x+y)^2_{|_X}$, then $\overline{\Sigma_{\mathrm{alg}} g} = \{\mathbf{0}\}$ and $\overline{\Sigma_{\mathrm{reg}} g} = \emptyset$; thus, in general, $\overline{\Sigma_{\mathrm{reg}} f} \neq \overline{\Sigma_{\mathrm{alg}} f}$.

While it is easy to produce examples where $\Sigma_{\mathrm{Nash}} f$ is not equal to $\Sigma_{\mathrm{alg}} f$ and examples where $\Sigma_{\mathrm{Nash}} f$ is not equal to $\Sigma_{\mathrm{cnr}} f$, it is not quite so easy to come up with examples where all three of these sets are distinct. We give such an example here.

Let $Z := V((y - zx)(y^2 - x^3)) \subseteq \mathbb{C}^3$ and $L := y_{|_Z}$. Then, one easily verifies that $\Sigma_{\mathrm{alg}} f = \emptyset$, $\Sigma_{\mathrm{Nash}} f = \{\mathbf{0}\}$, and $\Sigma_{\mathrm{cnr}} f = \mathbb{C} \times \{\mathbf{0}\}$.

If $X = V(xy)$ and $h := (x+y)_{|_X}$, then $\Sigma_{\mathbb{C}} h = \{\mathbf{0}\}$ and $\Sigma_{\mathrm{cnr}} h = \emptyset$; thus, in general, $\Sigma_{\mathrm{cnr}} f \neq \overline{\Sigma_{\mathbb{C}} f}$.

Let $W := V(z^5 + ty^6 z + y^7 x + x^{15}) \subseteq \mathbb{C}^4$; this is the example of Briançon and Speder [**B-S**] in which the topology along the t-axis is constant, despite the fact that the origin is a point-stratum in the canonical Whitney stratification of W. Hence, if we let r denote the restriction of t to W, then, for values of r close to 0, $\mathbf{0}$ is the only point in $\Sigma_{\mathrm{can}} r$ and $\mathbf{0} \notin \Sigma_{\mathbb{C}} r$. Therefore, $\mathbf{0} \in \Sigma_{\mathrm{can}} r \ - \ \overline{\Sigma_{\mathbb{C}} r}$, and so, in general, $\overline{\Sigma_{\mathbb{C}} f} \neq \Sigma_{\mathrm{can}} f$.

Using the coordinates (x, y, z) on $\mathbb{C}^3$, consider the cross-product $Y := V(y^2 - x^3) \subseteq \mathbb{C}^3$. The canonical Whitney stratification of Y is given by $\{Y - \{\mathbf{0}\} \times \mathbb{C}, \ \{\mathbf{0}\} \times \mathbb{C}\}$. Let $\pi := z_{|_Y}$. Then, $\Sigma_{\mathrm{can}} \pi = \emptyset$, while $\Sigma_{\mathrm{rdf}} \pi = \{\mathbf{0}\} \times \mathbb{C}$. Thus, in general, $\Sigma_{\mathrm{can}} f \neq \Sigma_{\mathrm{rdf}} f$.

It is, of course, easy to throw extra, non-canonical, Whitney strata into almost any example in order to see that, in general, $\Sigma_{\mathrm{can}} f \neq \Sigma_{\mathcal{S}} f$.

To summarize the contents of this example and Proposition 1.6: we have seven seemingly reasonable definitions of "critical locus" for complex analytic functions on singular spaces (we are not counting $\Sigma_{\mathcal{S}} f$, since it is not intrinsically defined). All of our critical locus avatars agree for manifolds. The sets $\Sigma_{\text{reg}}f$, $\Sigma_{\text{alg}}f$, $\Sigma_{\text{Nash}}f$, and $\Sigma_{\mathbb{C}}f$ need not be closed. There is a chain of containments among the closures of these critical loci, but – in general – none of the sets are equal.

However, we consider the sets $\overline{\Sigma_{\text{reg}}f}$, $\overline{\Sigma_{\text{alg}}f}$, $\overline{\Sigma_{\text{Nash}}f}$, and $\Sigma_{\text{cnr}}f$ to be too small; these "critical loci" do not detect the change in topology at the level hypersurface $h = 0$ in the simple example $X = V(xy)$ and $h = (x + y)_{|X}$ (from Example 1.8).

Despite the fact that the Stratified Morse Theory of [**Go-Mac1**] yields nice results and requires one to consider the stratified critical locus, we also will not use $\Sigma_{\text{can}}f$ (or any other $\Sigma_{\mathcal{S}}f$) as our primary notion of critical locus; $\Sigma_{\text{can}}f$ is often too big. As we saw in the Briançon-Speder example in Example 1.8, the stratified critical locus sometimes forces one to consider "critical points" which do not correspond to changes in topology.

Certainly, $\Sigma_{\text{rdf}}f$ is far too large, if we want critical points to have **any** relation to changes in the topology of level hypersurfaces: if X has a singular set ΣX, then the critical space of the projection $\pi : X \times \mathbb{C} \to \mathbb{C}$ would consist of $\Sigma X \times \mathbb{C}$, despite the obvious triviality of the family of level hypersurfaces defined by π.

Therefore, we choose to concentrate our attention on the $\mathbb{C}$-critical locus, and we will justify this choice with the results in the remainder of this paper.

Note that we consider $\Sigma_{\mathbb{C}}f$, not its closure, to be the correct notion of critical locus; we think that this is the more natural definition, and we consider the question of when $\Sigma_{\mathbb{C}}f$ is closed to be an interesting one. It is true, however, that all of our results refer to $\overline{\Sigma_{\mathbb{C}}f}$. We should mention here that, while $\Sigma_{\mathbb{C}}f$ need not be closed, the existence of Thom stratifications [**Hi**] implies that $\Sigma_{\mathbb{C}}f$ is at least analytically constructible; hence, $\overline{\Sigma_{\mathbb{C}}f}$ is an analytic subset of X.

Before we leave this section, in which we have already looked at seven definitions of "critical locus", we need to look at one last variant. As we mentioned at the end of the introduction, even though we wish to investigate the Milnor fibre with coefficients in $\mathbb{C}$, the fact that the shifted constant sheaf on a non-l.c.i. need not be perverse requires us to take the perverse cohomology of the constant sheaf. This means that we need to consider the hypercohomology of Milnor fibres with coefficients in an arbitrary bounded, constructible complex of sheaves (of $\mathbb{C}$-vector spaces).

The $\mathbb{C}-$critical locus is nicely described in terms of vanishing cycles (see [**K-S**] for general properties of vanishing cycles, but be aware that we use the more traditional shift):

$$\Sigma_{\mathbb{C}}f = \{\mathbf{x} \in X \mid H^*(\phi_{f-f(\mathbf{x})}\mathbb{C}^\bullet_X)_{\mathbf{x}} \neq 0\}.$$

This definition generalizes easily to yield a definition of the critical loci of f with respect to arbitrary bounded, constructible complexes of sheaves on X.

Let $\mathcal{S} := \{S_\alpha\}$ be a Whitney stratification of X, and let $\mathbf{F}^\bullet$ be a bounded complex of sheaves (of $\mathbb{C}$-vector spaces) which is constructible with respect to $\mathcal{S}$.

Definition 1.9. The $\mathbf{F}^\bullet$*-critical locus of* f , $\Sigma_{\mathbf{F}^\bullet}f$, is defined by

$$\Sigma_{\mathbf{F}^\bullet}f := \{\mathbf{x} \in X \mid H^*(\phi_{f-f(\mathbf{x})}\mathbf{F}^\bullet)_{\mathbf{x}} \neq 0\}.$$

Remark 1.10. Stratified Morse Theory (see [**Go-Mac1**]) implies that $\Sigma_{\mathbf{F}^\bullet}f \subseteq \Sigma_{\mathcal{S}}f$ (alternatively, this follows from 8.4.1 and 8.6.12 of [**K-S**], combined with the facts that complex analytic Whitney stratifications are w-stratifications, and w-stratifications are μ-stratifications.)

We could discuss three more notions of the critical locus of a function – two of which are obtained by picking specific complexes for $\mathbf{F}^\bullet$ in Definition 1.9. However, we will defer the introduction of these new critical loci until Section 3; at that point, we will have developed the tools necessary to say something interesting about these three new definitions.

Section 2. THE LINK BETWEEN THE ALGEBRAIC AND TOPOLOGICAL POINTS OF VIEW.

We continue with our previous notation: X is a complex analytic space contained in some open subset $\mathcal{U}$ of some $\mathbb{C}^{n+1}$, $\tilde{f}: \mathcal{U} \to \mathbb{C}$ is a complex analytic function, $f = \tilde{f}_{|_X}$, $\mathcal{S} = \{S_\alpha\}$ is a Whitney stratification of X with connected strata, and $\mathbf{F}^\bullet$ is a bounded complex of sheaves (of $\mathbb{C}$-vector spaces) which is constructible with respect to $\mathcal{S}$. In addition, N_α and $\mathbb{L}_\alpha$ are, respectively, the normal slice and complex link of the d_α-dimensional stratum S_α (see [**Go-Mac1**]).

In this section, we are going to prove a general result which describes the characteristic cycle of $\phi_f\mathbf{F}^\bullet$ in terms of blowing-up the image of $d\tilde{f}$ inside the conormal spaces to strata. We will have to wait until the next section (on results for perverse sheaves) to actually show how this provides a relationship between $\Sigma_{\mathbf{F}^\bullet} f$ and $\Sigma_{\mathcal{S}} f$ in the case where $\mathbf{F}^\bullet$ is perverse.

Beginning in this section, we will use some aspects of intersection theory, as described in [**F**]; however, at all times, the setting for our intersections will be the most trivial: we will only consider proper intersections of complex analytic cycles (not cycle classes) inside an ambient analytic manifold. In this setting, there is a well-defined intersection cycle.

Definition 2.1. Recall that the *characteristic cycle,* Ch($\mathbf{F}^\bullet$)*, of* $\mathbf{F}^\bullet$ in $T^*\mathcal{U}$ is the linear combination $\sum_\alpha m_\alpha(\mathbf{F}^\bullet)\left[\overline{T^*_{S_\alpha}\mathcal{U}}\right]$, where the $m_\alpha(\mathbf{F}^\bullet)$ are integers given by

$$m_\alpha(\mathbf{F}^\bullet) \ := \ (-1)^{\dim X-1}\chi(\phi_{L_{|_X}}\mathbf{F}^\bullet)_\mathbf{x} \ = \ (-1)^{\dim X-d_\alpha-1}\chi(\phi_{L_{|_{N_\alpha}}}\mathbf{F}^\bullet{}_{|_{N_\alpha}})_\mathbf{x}$$

for any point $\mathbf{x}$ in S_α, with normal slice N_α at $\mathbf{x}$, and any $L : (\mathcal{U}, x) \to \ (\mathbb{C}, 0)$ such that $d_\mathbf{x}L$ is a non-degenerate covector at $\mathbf{x}$ (with respect to our fixed stratification; see [**Go-Mac1**]) and $L_{|_{S_\alpha}}$ has a Morse singularity at $\mathbf{x}$. This cycle is independent of all the choices made (see, for instance, [**K-S**, Chapter IX]).

We need a number of preliminary results before we can prove the main theorem (Theorem 2.10) of this section.

Definition 2.2. Recall that, if M is an analytic submanifold of $\mathcal{U}$ and $M \subseteq X$, then the *relative conormal space (of* M *with respect to* f *in* $\mathcal{U}$*)*, $T^*_{f_{|_M}}\mathcal{U}$, is given by

$$T^*_{f_{|_M}}\mathcal{U} := \{(\mathbf{x}, \eta) \in T^*\mathcal{U} \mid \mathbf{x} \in M,\ \eta\big(\ker d_\mathbf{x}(f_{|_M})\big) = 0\} =$$
$$\{(\mathbf{x}, \eta) \in T^*\mathcal{U} \mid \mathbf{x} \in M,\ \eta\big(T_\mathbf{x}M \cap \ker d_\mathbf{x}\tilde{f}\big) = 0\}.$$

We define the *total relative conormal cycle*, $T^*_{f,\mathbf{F}^\bullet}\mathcal{U}$, by $T^*_{f,\mathbf{F}^\bullet}\mathcal{U} := \displaystyle\sum_{S_\alpha \not\subseteq f^{-1}(0)} m_\alpha \left[\overline{T^*_{f_{|_{S_\alpha}}}\mathcal{U}}\right]$.

From this point, through Lemma 2.9, it will be convenient to assume that we have refined

our stratification $\mathcal{S} = \{S_\alpha\}$ so that $V(f)$ is a union of strata. By Remark 1.7, this implies that, in a neighborhood of $V(f)$, if $S_\alpha \not\subseteq V(f)$, then $\Sigma(f_{|_{S_\alpha}}) = \emptyset$.

We shall need the following important result from [**BMM**, 3.4.2].

Theorem 2.3. ([**BMM**]) *The characteristic cycle of the sheaf of nearby cycles of* $\mathbf{F}^\bullet$ *along* f, $\mathrm{Ch}\left(\psi_f \mathbf{F}^\bullet\right)$, *is isomorphic to the intersection product* $T^*_{f,\mathbf{F}^\bullet}\mathcal{U} \cdot \left(V(f) \times \mathbb{C}^{n+1}\right)$ *in* $\mathcal{U} \times \mathbb{C}^{n+1}$.

Let $\Gamma^1_{f,L}(S_\alpha)$ denote the closure in X of the relative polar curve of f with respect to L inside S_α (see [**M1**] and [**M3**]). It is important to note that $\Gamma^1_{f,L}(S_\alpha)$ is the closure of the polar curve in S_α, **not** in $\overline{S_\alpha}$; that is, $\Gamma^1_{f,L}(S_\alpha)$ has no components contained in any strata $S_\beta \subseteq \overline{S_\alpha}$ such that $S_\beta \neq S_\alpha$.

It is convenient to have a specific point in X at which to work. Below, we concentrate our attention at the origin; of course, if the origin is not in X (or, if the origin is not in $V(f)$), then we obtain zeroes for all the terms below. For any bounded, constructible complex $\mathbf{A}^\bullet$ on a subspace of $\mathcal{U}$, let $m_{\mathbf{0}}(\mathbf{A}^\bullet)$ equal the coefficient of $\left[T^*_{\{\mathbf{0}\}}\mathcal{U}\right]$ in the characteristic cycle of $\mathbf{A}^\bullet$.

We need to state one further result without proof – this result can be obtained from [**BMM**], but we give the result as stated in [**M1**, 4.6].

Theorem 2.4. *For generic linear forms* L, *we have the following formulas:*

$$m_{\mathbf{0}}(\psi_f \mathbf{F}^\bullet) = \sum_{S_\alpha \not\subseteq V(f)} m_\alpha \left(\Gamma^1_{f,L}(S_\alpha) \cdot V(f)\right)_{\mathbf{0}};$$

$$m_{\mathbf{0}}(\mathbf{F}^\bullet) + m_{\mathbf{0}}(\mathbf{F}^\bullet_{|_{V(f)}}) = \sum_{S_\alpha \not\subseteq V(f)} m_\alpha \left(\Gamma^1_{f,L}(S_\alpha) \cdot V(L)\right)_{\mathbf{0}}; \text{ and}$$

$$m_{\mathbf{0}}(\phi_f \mathbf{F}^\bullet) = m_{\mathbf{0}}(\mathbf{F}^\bullet) + \sum_{S_\alpha \not\subseteq V(f)} m_\alpha \left(\left(\Gamma^1_{f,L}(S_\alpha) \cdot V(f)\right)_{\mathbf{0}} - \left(\Gamma^1_{f,L}(S_\alpha) \cdot V(L)\right)_{\mathbf{0}}\right).$$

Lemma 2.5. *If* $S_\alpha \not\subseteq f^{-1}(0)$, *then the coefficient of* $\left[\mathbb{P}(T^*_{\{\mathbf{0}\}}\mathcal{U})\right]$ *in* $\mathbb{P}\left(\overline{T^*_{f_{|_{S_\alpha}}}\mathcal{U}}\right) \cdot \left(V(f) \times \mathbb{P}^n\right)$ *is given by* $\left(\Gamma^1_{f,L}(S_\alpha) \cdot V(f)\right)_{\mathbf{0}}$.

Proof. Take a complex of sheaves, $\mathbf{F}^\bullet$, which has a characteristic cycle consisting only of $\left[\overline{T^*_{S_\alpha}\mathcal{U}}\right]$ (see, for instance, [**M1**]). Now, apply the formula for $m_{\mathbf{0}}(\psi_f \mathbf{F}^\bullet)$ from Theorem 2.4 together with Theorem 2.3. □

We need to establish some notation that we shall use throughout the remainder of this section.

Using the isomorphism, $T^*\mathcal{U} \cong \mathcal{U} \times \mathbb{C}^{n+1}$, we consider $\mathrm{Ch}(\mathbf{F}^\bullet)$ as a cycle in $X \times \mathbb{C}^{n+1}$; we use $\mathbf{z} := (z_0, \dots, z_n)$ as coordinates on $\mathcal{U}$ and $\mathbf{w} := (w_0, \dots, w_n)$ as the cotangent coordinates.

Let I denote the sheaf of ideals on $\mathcal{U}$ given by the image of $d\tilde{f}$, i.e., $I = \langle w_0 - \frac{\partial \tilde{f}}{\partial z_0}, \dots, w_n - \frac{\partial \tilde{f}}{\partial z_n} \rangle$. For all α, let $B_\alpha = \mathrm{Bl}_{\mathrm{im}\, d\tilde{f}}\, \overline{T^*_{S_\alpha}\mathcal{U}}$ denote the blow-up of $\overline{T^*_{S_\alpha}\mathcal{U}}$ along the image of I in $\overline{T^*_{S_\alpha}\mathcal{U}}$, and let E_α denote the corresponding exceptional divisor. For all α, we have $E_\alpha \subseteq B_\alpha \subseteq X \times \mathbb{C}^{n+1} \times \mathbb{P}^n$. Let $\pi : X \times \mathbb{C}^{n+1} \times \mathbb{P}^n \to X \times \mathbb{P}^n$ denote the projection. Note that, if $(\mathbf{x}, \mathbf{w}, [\eta]) \in E_\alpha$, then $\mathbf{w} = d_{\mathbf{x}}\tilde{f}$ and so, for all α, π induces an isomorphism from E_α to $\pi(E_\alpha)$. We refer to $E := \sum_\alpha m_\alpha E_\alpha$ as the *total exceptional divisor* inside the total blow-up $\mathrm{Bl}_{\mathrm{im}\, d\tilde{f}}\, \mathrm{Ch}(\mathbf{F}^\bullet) := \sum_\alpha m_\alpha \mathrm{Bl}_{\mathrm{im}\, d\tilde{f}} \left[\overline{T^*_{S_\alpha}\mathcal{U}}\right]$.

Lemma 2.6. *For all S_α, there is an inclusion $\pi\left(\mathrm{Bl}_{\mathrm{im}\, d\tilde{f}}\, \overline{T^*_{S_\alpha}\mathcal{U}}\right) \subseteq \mathbb{P}\big(\overline{T^*_{f_{|_{S_\alpha}}}\mathcal{U}}\big)$.*

Proof. This is entirely straightforward. Suppose that

$$(\mathbf{x}, \mathbf{w}, [\eta]) \in \mathrm{Bl}_{\mathrm{im}\, d\tilde{f}}\, \overline{T^*_{S_\alpha}\mathcal{U}} = \overline{\mathrm{Bl}_{\mathrm{im}\, d\tilde{f}}\, T^*_{S_\alpha}\mathcal{U}}.$$

Then, we have a sequence $(\mathbf{x}_i, \mathbf{w}_i, [\eta_i]) \in \mathrm{Bl}_{\mathrm{im}\, d\tilde{f}}\, T^*_{S_\alpha}\mathcal{U}$ such that $(\mathbf{x}_i, \mathbf{w}_i, [\eta_i]) \to (\mathbf{x}, \mathbf{w}, [\eta])$.

By definition of the blow-up, for each $(\mathbf{x}_i, \mathbf{w}_i, [\eta_i])$, there exists a sequence $(\mathbf{x}_i^j, \mathbf{w}_i^j) \in T^*_{S_\alpha}\mathcal{U} - \mathrm{im}\, d\tilde{f}$ such that $(\mathbf{x}_i^j, \mathbf{w}_i^j, [\mathbf{w}_i^j - d_{\mathbf{x}_i^j}\tilde{f}]) \to (\mathbf{x}_i, \mathbf{w}_i, [\eta_i])$. Now, $(\mathbf{x}_i^j, [\mathbf{w}_i^j - d_{\mathbf{x}_i^j}\tilde{f}])$ is clearly in $\mathbb{P}\big(T^*_{f_{|_{S_\alpha}}}\mathcal{U}\big)$, and so each $(\mathbf{x}_i, [\eta_i])$ is in $\mathbb{P}\big(\overline{T^*_{f_{|_{S_\alpha}}}\mathcal{U}}\big)$. Therefore, $(\mathbf{x}, [\eta]) \in \mathbb{P}\big(\overline{T^*_{f_{|_{S_\alpha}}}\mathcal{U}}\big)$. □

Lemma 2.7. *If $S_\alpha \not\subseteq f^{-1}(0)$, then the coefficient of $\left[\mathbb{P}\big(T^*_{\{\mathbf{0}\}}\mathcal{U}\big)\right] = \{\mathbf{0}\} \times \mathbb{P}^n$ in $\pi_*(E_\alpha)$ equals $\big(\Gamma^1_{f,L}(S_\alpha) \cdot V(f)\big)_{\mathbf{0}} - \big(\Gamma^1_{f,L}(S_\alpha) \cdot V(L)\big)_{\mathbf{0}}$.*

Proof. We will work inside $\mathcal{U} \times \mathbb{C}^{n+1} \times \mathbb{P}^n$. We use $[u_0 : \dots : u_n]$ as projective coordinates, and calculate the coefficient of $G_{\mathbf{0}} := \left[\{\mathbf{0}\} \times \{d_{\mathbf{0}}\tilde{f}\} \times \mathbb{P}^n\right]$ in E_α using the affine patch $\{u_0 \neq 0\}$.

Letting $\tilde{u}_i = u_i/u_0$ for $i \geqslant 1$, we have $\{u_0 \neq 0\} \cap B_\alpha =$

$$\overline{\left\{ \big(\mathbf{x}, \mathbf{w}, (\tilde{u}_1, \dots, \tilde{u}_n)\big) \in \big(\overline{T^*_{S_\alpha}\mathcal{U}} - \mathrm{im}\, d\tilde{f}\big) \times \mathbb{C}^n \;\middle|\; w_i - \frac{\partial \tilde{f}}{\partial z_i} = \tilde{u}_i \left(w_0 - \frac{\partial \tilde{f}}{\partial z_0}\right),\ i \geqslant 1 \right\}},$$

and $\{u_0 \neq 0\} \cap E_\alpha$ equals the intersection product $\big(\{u_0 \neq 0\} \cap B_\alpha\big) \cdot V\left(w_0 - \frac{\partial \tilde{f}}{\partial z_0}\right)$ in $\mathcal{U} \times \mathbb{C}^{n+1} \times \mathbb{C}^n$.

To calculate the multiplicity of $\{u_0 \neq 0\} \cap G_{\mathbf{0}}$ in $\left(\{u_0 \neq 0\} \cap B_\alpha\right) \cdot V\left(w_0 - \frac{\partial \tilde{f}}{\partial z_0}\right)$, we move to a generic point of $\{u_0 \neq 0\} \cap G_{\mathbf{0}}$ and take a normal slice; that is, we fix a generic choice $(\tilde{u}_1, \dots, \tilde{u}_n) = (a_1, \dots, a_n)$. This corresponds to choosing the generic linear form $L = z_0 + a_1 z_1 + \dots a_n z_n$.

We claim that $Z := \{u_0 \neq 0\} \cap B_\alpha \cap V(\tilde{u}_1 - a_1, \dots, \tilde{u}_n - a_n) - \{\mathbf{0}\} \times \mathbb{C}^{n+1} \times \mathbb{C}^n$ equals the set of all $\big(\mathbf{x}, \mathbf{w}, (a_1, \dots, a_n)\big)$ such that $\mathbf{x} \in \Gamma^1_{f,L}(S_\alpha) - \{\mathbf{0}\}$ and $\mathbf{w} = d_{\mathbf{x}}\tilde{f} - \lambda(\mathbf{x}) d_{\mathbf{x}}L$, where $\lambda(\mathbf{x})$ is the unique non-zero complex number such that $\big(d_{\mathbf{x}}\tilde{f} - \lambda(\mathbf{x}) d_{\mathbf{x}}L\big)(T_{\mathbf{x}}S_\alpha) = 0$.

Once we show that $\mathbf{x}$ must be in $\Gamma^1_{f,L}(S_\alpha) - \{\mathbf{0}\}$, then it follows at once from the definition of the relative polar curve that there exists a $\lambda(\mathbf{x})$ as above. That such a $\lambda(\mathbf{x})$ must be unique is easy: if we had two distinct such λ, then we would have $d_{\mathbf{x}}L(T_{\mathbf{x}}S_\alpha) = 0$ – but this is impossible for generic L.

Now, by definition of the relative polar curve and using 2.6, we find

$$\pi\big(\{u_0 \neq 0\} \cap B_\alpha \cap V(\tilde{u}_1 - a_1, \dots, \tilde{u}_n - a_n) - \{\mathbf{0}\} \times \mathbb{C}^{n+1} \times \mathbb{C}^n\big)$$

$$\subseteq \{u_0 \neq 0\} \cap \mathbb{P}\big(\overline{T^*_{f_{|S_\alpha}}\mathcal{U}}\big) \cap V(\tilde{u}_1 - a_1, \dots, \tilde{u}_n - a_n) - \{\mathbf{0}\} \times \mathbb{C}^{n+1} \times \mathbb{C}^n$$
$$= \big(\Gamma^1_{f,L}(S_\alpha) - \{\mathbf{0}\}\big) \times \{(a_1, \dots, a_n)\}.$$

(Actually, here we have also used 2.4 and 2.5 to conclude that there are no components of the relative polar curve which are contained in strata other than S_α.)

Thus,

$$Z = \big\{(\mathbf{x}, \mathbf{w}, (a_1, \dots, a_n)) \mid \mathbf{x} \in \big(\Gamma^1_{f,L}(S_\alpha) - \{\mathbf{0}\}\big) \text{ and } \mathbf{w} = d_{\mathbf{x}}\tilde{f} - \lambda(\mathbf{x}) d_{\mathbf{x}} L\big\},$$

and the coefficient of $G_{\mathbf{0}}$ in E_α equals the intersection number

$$\left(\overline{Z} \cdot V\Big(w_0 - \frac{\partial \tilde{f}}{\partial z_0}\Big)\right)_{(\mathbf{0}, \mathbf{0}, (a_1, \dots, a_n))}$$

in $\mathcal{U} \times \mathbb{C}^{n+1} \times \{(a_1, \dots, a_n)\}$.

Now, for each component C of $\overline{Z}$ through $(\mathbf{0}, \mathbf{0}, (a_1, \dots, a_n))$, select a local analytic parameterization $\mathbf{u}_C(t) = (\mathbf{x}_C(t), \mathbf{w}_C(t), (a_1, \dots, a_n)) \in C$ such that $\mathbf{x}_C(0) = \mathbf{0}$, $\mathbf{w}_C(0) = \mathbf{0}$, and, for $t \neq 0$, $\mathbf{u}_C(t) \in C - \{(\mathbf{0}, \mathbf{0})\} \times \mathbb{C}^n$. Then,

$$\left(\overline{Z} \cdot V\Big(w_0 - \frac{\partial \tilde{f}}{\partial z_0}\Big)\right)_{(\mathbf{0}, \mathbf{0}, (a_1, \dots, a_n))} = \sum_C \operatorname{mult}\left\{\Big(w_0 - \frac{\partial \tilde{f}}{\partial z_0}\Big) \circ \mathbf{u}_C(t)\right\}.$$

Moreover, a quick look at the definition of Z tells us that $\Big(w_0 - \frac{\partial \tilde{f}}{\partial z_0}\Big) \circ \mathbf{u}_C(t) = \lambda(\mathbf{x}_C(t))$. Thus, what we want to show is that

$$\sum_C \operatorname{mult} \lambda(\mathbf{x}_C(t)) = \big(\Gamma^1_{f,L}(S_\alpha) \cdot V(f)\big)_{\mathbf{0}} - \big(\Gamma^1_{f,L}(S_\alpha) \cdot V(L)\big)_{\mathbf{0}}.$$

If we look now at $\big(\Gamma^1_{f,L}(S_\alpha) \cdot V(f)\big)_{\mathbf{0}}$, we find

$$\big(\Gamma^1_{f,L}(S_\alpha) \cdot V(f)\big)_{\mathbf{0}} = \sum_C \operatorname{mult} f(\mathbf{x}_C(t)) = \sum_C \Big(1 + \operatorname{mult}\big(f(\mathbf{x}_C(t))\big)'\Big)$$
$$= \sum_C \Big(1 + \operatorname{mult} d_{\mathbf{x}_C(t)} \tilde{f}(\mathbf{x}'_C(t))\Big) = \sum_C \Big(1 + \operatorname{mult}\big((\mathbf{w}_C(t) + \lambda(\mathbf{x}_C(t)) d_{\mathbf{x}_C(t)} L) \circ (\mathbf{x}'_C(t))\big)\Big).$$

As $(\mathbf{x}_C(t), \mathbf{w}_C(t)) \in T^*_{S_\alpha}\mathcal{U}$ for $t \neq 0$, $\mathbf{w}_C(t) \circ \mathbf{x}'_C(t) = 0$. In addition, $d_{\mathbf{x}_C(t)} L \circ \mathbf{x}'_C(t) = \big(L(\mathbf{x}_C(t))\big)'$, and so we obtain that

$$\big(\Gamma^1_{f,L}(S_\alpha) \cdot V(f)\big)_{\mathbf{0}} =$$
$$\sum_C \big(\operatorname{mult} L(p_C(t)) + \operatorname{mult} \lambda(p_C(t))\big) = \big(\Gamma^1_{f,L}(S_\alpha) \cdot V(L)\big)_{\mathbf{0}} + \sum_C \operatorname{mult} \lambda(\mathbf{x}_C(t)),$$

and so we are finished. $\square$

Lemma 2.8. *For all α such that $S_\alpha \subseteq V(f)$, there is an inclusion of the exceptional divisor*

$$E_\alpha \cong \pi(E_\alpha) \subseteq \mathbb{P}\big(\overline{T^*_{f_{|S_\alpha}}\mathcal{U}}\big) \cap \big(V(f) \times \mathbb{P}^n\big).$$

Proof. That π is an isomorphism when restricted to the exceptional divisor is trivial: $(\mathbf{x}, \mathbf{w}, [\eta]) \in E_\alpha$ implies that $\mathbf{w} = d_{\mathbf{x}}\tilde{f}$. From Lemma 2.6, $\pi(E_\alpha) \subseteq \pi\left(\mathrm{Bl}_{\mathrm{im}\, d\tilde{f}}\, \overline{T^*_{S_\alpha}\mathcal{U}}\right) \subseteq \mathbb{P}\big(\overline{T^*_{f_{|S_\alpha}}\mathcal{U}}\big)$. The result follows. $\square$

Lemma 2.9. *If $S_\alpha \subseteq f^{-1}(0)$, then $E_\alpha \cong \pi(E_\alpha) = \mathbb{P}(\overline{T^*_{S_\alpha}\mathcal{U}})$.*

Proof. If $S_\alpha \subseteq f^{-1}(0)$, then $\mathbb{P}\big(T^*_{f_{|S_\alpha}}\mathcal{U}\big) = \mathbb{P}\big(T^*_{S_\alpha}\mathcal{U}\big)$, and so, by 2.8, $\pi(E_\alpha) \subseteq \mathbb{P}(\overline{T^*_{S_\alpha}\mathcal{U}})$. We will demonstrate the reverse inclusion.

Suppose that we have $(\mathbf{x}, [\eta]) \in \mathbb{P}(\overline{T^*_{S_\alpha}\mathcal{U}})$. Then, there exists a sequence $(\mathbf{x}_i, \eta_i) \in T^*_{S_\alpha}\mathcal{U}$ such that $(\mathbf{x}_i, \eta_i) \to (\mathbf{x}, \eta)$. Hence, $\big(\mathbf{x}_i,\ \frac{1}{i}\eta_i + d_{\mathbf{x}_i}\tilde{f}\big) \in \overline{T^*_{S_\alpha}\mathcal{U}} - \mathrm{im}\, d\tilde{f}$ and

$$\left(\mathbf{x}_i,\ \frac{1}{i}\eta_i + d_{\mathbf{x}_i}\tilde{f},\ \left[\left(\frac{1}{i}\eta_i + d_{\mathbf{x}_i}\tilde{f}\right) - d_{\mathbf{x}_i}\tilde{f}\right]\right) \to (\mathbf{x}, d_{\mathbf{x}}\tilde{f}, [\eta]) \in E_\alpha. \quad \square$$

We come now to the main theorem of this section. This theorem relates the topological data provided by the vanishing cycles of a function f to the algebraic data given by blowing-up the image of the differential of an extension of f.

Theorem 2.10. *The projection π induces an isomorphism between the total exceptional divisor $E \subseteq \mathrm{Bl}_{\mathrm{im}\, d\tilde{f}}\, \mathrm{Ch}(\mathbf{F}^\bullet)$ and the sum over all $v \in \mathbb{C}$ of the projectivized characteristic cycles of the sheaves of vanishing cycles of $\mathbf{F}^\bullet$ along $f - v$, i.e.,*

$$E \cong \pi_*(E) = \sum_{v \in \mathbb{C}} \mathbb{P}(\mathrm{Ch}(\phi_{f-v}\mathbf{F}^\bullet)).$$

Proof. Remarks 1.7 and 1.10 imply that, locally, $\mathrm{supp}\, \phi_{f-v}\mathbf{F}^\bullet \subseteq f^{-1}(v)$. As the $\mathbb{P}(\mathrm{Ch}(\phi_{f-v}\mathbf{F}^\bullet))$ are disjoint for different values of v, we may immediately reduce ourselves to the case where we are working near $\mathbf{0} \in X$ and where $f(\mathbf{0}) = 0$. We refine our stratification so that, for all α, $\Sigma(f_{|S_\alpha}) = \emptyset$ unless $S_\alpha \subseteq V(f)$. As any newly introduced stratum will appear with a coefficient of zero in the characteristic cycle, the total exceptional divisor will not change. We need to show that $E \cong \pi(E) = \mathbb{P}(\mathrm{Ch}(\phi_f\mathbf{F}^\bullet))$.

Now, we will first show that $\pi(E)$ is Lagrangian.

If $S_\alpha \subseteq f^{-1}(0)$, then $\pi(E_\alpha) = \mathbb{P}(\overline{T^*_{S_\alpha}\mathcal{U}})$ by 2.9. If $S_\alpha \not\subseteq f^{-1}(0)$, then, by Theorem 2.3, $\mathbb{P}\big(\overline{T^*_{f_{|S_\alpha}}\mathcal{U}}\big) \cap \big(V(f) \times \mathbb{P}^n\big)$ is Lagrangian and, in particular, is purely n-dimensional. By Lemma 2.8, $\pi(E_\alpha)$ is a purely n-dimensional analytic set contained in $\mathbb{P}\big(\overline{T^*_{f_{|S_\alpha}}\mathcal{U}}\big) \cap \big(V(f) \times \mathbb{P}^n\big)$. We need to show that $\pi(E_\alpha)$ is closed.

Suppose we have a sequence $(\mathbf{x}_i, [\eta_i]) \in \pi(E_\alpha)$ and $(\mathbf{x}_i, [\eta_i]) \to (\mathbf{x}, [\eta])$ in $\mathcal{U} \times \mathbb{P}^n$. Then, there exists a sequence $\mathbf{w}_i$ so that $(\mathbf{x}_i, \mathbf{w}_i, [\eta_i]) \in E_\alpha$; by definition of the exceptional divisor, this implies $\mathbf{w}_i = d_{\mathbf{x}_i}\tilde{f}$. Therefore, $(\mathbf{x}_i, \mathbf{w}_i, [\eta_i]) \to (\mathbf{x}, d_{\mathbf{x}}\tilde{f}, [\eta])$, which is contained in E_α since E_α is closed in $\mathcal{U} \times \mathbb{C}^{n+1} \times \mathbb{P}^n$. Thus, $(\mathbf{x}, [\eta]) \in \pi(E_\alpha)$, and so $\pi(E_\alpha)$ is closed and, hence, Lagrangian.

Now, $\pi(E)$ and $\mathbb{P}(\mathrm{Ch}(\phi_f\mathbf{F}^\bullet))$ are both supported over $\Sigma_{\mathcal{S}} f$ and, by taking normal slices to strata, we are reduced to the point-stratum case. Thus, what we need to show is: the coefficient of $\big[\mathbb{P}\big(T^*_{\{\mathbf{0}\}}\mathcal{U}\big)\big]$ in

E equals the coefficient of $\left[\mathbb{P}\left(T^*_{\{\mathbf{0}\}}\mathcal{U}\right)\right]$ in $\mathbb{P}(\mathrm{Ch}(\phi_f\mathbf{F}^\bullet))$. Using 2.4, this is equivalent to showing that the coefficient of $\left[\mathbb{P}\left(T^*_{\{\mathbf{0}\}}\mathcal{U}\right)\right]$ in E equals

$$m_{\mathbf{0}}(\mathbf{F}^\bullet) + \sum_{S_\alpha \not\subseteq V(f)} m_\alpha \left(\left(\Gamma^1_{f,L}(S_\alpha)\cdot V(f)\right)_{\mathbf{0}} - \left(\Gamma^1_{f,L}(S_\alpha)\cdot V(L)\right)_{\mathbf{0}}\right)$$

for a generic linear form L.

But, by 2.9,

$$E \;=\; \sum_\alpha m_\alpha E_\alpha \;=\; \sum_{S_\alpha \subseteq V(f)} m_\alpha \left[\mathbb{P}(\overline{T^*_{S_\alpha}\mathcal{U}})\right] \;+\; \sum_{S_\alpha \not\subseteq V(f)} m_\alpha E_\alpha$$

and the coefficient of $\left[\mathbb{P}\left(T^*_{\{\mathbf{0}\}}\mathcal{U}\right)\right]$ in $\displaystyle\sum_{S_\alpha \subseteq V(f)} m_\alpha \left[\mathbb{P}(\overline{T^*_{S_\alpha}\mathcal{U}})\right]$ is precisely $m_{\mathbf{0}}(\mathbf{F}^\bullet)$.

Therefore, we will be finished if we can show that the coefficient of $\left[\mathbb{P}\left(T^*_{\{\mathbf{0}\}}\mathcal{U}\right)\right]$ in E_α equals $\left(\Gamma^1_{f,L}(S_\alpha)\cdot V(f)\right)_{\mathbf{0}} - \left(\Gamma^1_{f,L}(S_\alpha)\cdot V(L)\right)_{\mathbf{0}}$ if $S_\alpha \not\subseteq V(f)$. However, this is exactly the content of Lemma 2.7. □

Remark 2.11. In special cases, Theorem 2.10 was already known.

Consider the case where $X = \mathcal{U}$ and $\mathbf{F}^\bullet$ is the constant sheaf. Then, $\mathrm{Ch}(\mathbf{F}^\bullet) = \mathcal{U}\times\{\mathbf{0}\}$, and the image of $d\tilde{f}$ in $\mathcal{U}\times\{\mathbf{0}\}$ is simply defined by the Jacobian ideal of f. Hence, our result reduces to the result obtained from the work of Kashiwara in [**K**] and Lê-Mebkhout in [**L-M**] – namely, that the projectivized characteristic cycle of the sheaf of vanishing cycles is isomorphic to the exceptional divisor of the blow-up of the Jacobian ideal in affine space.

As a second special case, suppose that X and $\mathbf{F}^\bullet$ are completely general, but that $\mathbf{x}$ is an isolated point in the image of $\mathrm{Ch}(\phi_f\mathbf{F}^\bullet)$ in X (for instance, $\mathbf{x}$ might be an isolated point in $\operatorname{supp}\phi_f\mathbf{F}^\bullet$). Then, for every stratum for which $m_\alpha \neq 0$, $(\mathbf{x}, d_{\mathbf{x}}\tilde{f})$ is an isolated point of $\operatorname{im} d\tilde{f}\cap\overline{T^*_{S_\alpha}\mathcal{U}}$ or is not contained in the intersection at all. Now, $\overline{T^*_{S_\alpha}\mathcal{U}}$ is an $(n+1)$-dimensional analytic variety and $\operatorname{im} d\tilde{f}$ is defined by $n+1$ equations. Therefore, $(\mathbf{x}, d_{\mathbf{x}}\tilde{f})$ is regularly embedded in $\overline{T^*_{S_\alpha}\mathcal{U}}$.

It follows that the exceptional divisor of the blow-up of $\operatorname{im} d\tilde{f}$ in $\overline{T^*_{S_\alpha}\mathcal{U}}$ has one component over $(\mathbf{x}, d_{\mathbf{x}}\tilde{f})$ and that that component occurs with multiplicity precisely equal to the intersection multiplicity $\left(\operatorname{im} d\tilde{f}\cdot\overline{T^*_{S_\alpha}\mathcal{U}}\right)_{(\mathbf{x}, d_{\mathbf{x}}\tilde{f})}$ in $T^*\mathcal{U}$. Thus, we recover the results of three independent works appearing in [**G**], [**Lê**], and [**S**] – that the coefficient of $\{\mathbf{x}\}\times\mathbb{C}^{n+1}$ in $\mathrm{Ch}(\phi_f\mathbf{F}^\bullet)$ is given by $\left(\operatorname{im} d\tilde{f}\cdot\mathrm{Ch}(\mathbf{F}^\bullet)\right)_{(\mathbf{x}, d_{\mathbf{x}}\tilde{f})}$.

In addition to generalizing the above results, Theorem 2.10 fits in well with Theorem 3.4.2 of [**BMM**]; that theorem contains a nice description of the characteristic cycles of the nearby cycles and of the restriction of a complex to a hypersurface. However, [**BMM**] does not contain a nice description of the vanishing cycles, nor does our Theorem 2.10 seem to follow easily from the results of [**BMM**]; in fact, Example 3.4.3 of [**BMM**] makes it clear that the general result contained in our Theorem 2.10 was unknown – for Briançon, Maisonobe, and Merle only derive the vanishing cycle result from their nearby cycle result in the easy, known case where the vanishing cycles are supported on an isolated point and, even then, they must make half a page of argument.

Corollary 2.12. *For each extension* $\tilde{f}$ *of* f*, let* $E_{\tilde{f}}$ *denote the exceptional divisor in* $\mathrm{Bl}_{\operatorname{im} d\tilde{f}}\overline{T^*_{X_{\mathrm{reg}}}\mathcal{U}}$*. Then,* $\pi\left(E_{\tilde{f}}\right)$ *is independent of* $\tilde{f}$*.*

Proof. We apply Theorem 2.10 to a complex of sheaves $\mathbf{F}^\bullet$ such that $m_\alpha = 1$ for each smooth component of $X_{\rm reg}$ and $m_\alpha = 0$ for every other stratum in some Whitney stratification of X (it is easy to produce such an $\mathbf{F}^\bullet$ – see, for instance, Lemma 3.1 of [**M1**]). The corollary follows from the fact that $\mathbb{P}(\mathrm{Ch}(\phi_f\mathbf{F}^\bullet))$ does not depend on the extension. □

Section 3. THE SPECIAL CASE OF PERVERSE SHEAVES.

We continue with our previous notation.

For the purposes of this paper, perverse sheaves are important because the vanishing cycles functor (shifted by -1) applied to a perverse sheaf once again yields a perverse sheaf and because of the following lemma.

Lemma 3.1. *If $\mathbf{P}^\bullet$ is a perverse sheaf on X, then $\mathrm{Ch}(\mathbf{P}^\bullet) = \sum_\alpha m_\alpha \left[\overline{T^*_{S_\alpha}\mathcal{U}}\right]$, where*

$$m_\alpha = (-1)^{\dim X} \dim H^0(N_\alpha, \mathbb{L}_\alpha;\ \mathbf{P}^\bullet_{|_{N_\alpha}}[-d_\alpha]);$$

in particular, $(-1)^{\dim X}\,\mathrm{Ch}(\mathbf{P}^\bullet)$ is a non-negative cycle.

If $\mathbf{P}^\bullet$ is perverse on X (or, even, perverse up to a shift), then $\mathrm{supp}\,\mathbf{P}^\bullet$ *equals the image in X of the characteristic cycle of $\mathbf{P}^\bullet$.*

Proof. The first statement follows from the definition of the characteristic cycle, together with the fact that a perverse sheaf supported on a point has non-zero cohomology only in degree zero.

The second statement follows at once from the fact that if $\mathbf{P}^\bullet$ is perverse up to a shift, then so is the restriction of $\mathbf{P}^\bullet$ to its support. Hence, by the support condition on perverse sheaves, there is an open dense set of the support, Ω, such that, for all $\mathbf{x} \in \Omega$, $H^*(\mathbf{P}^\bullet)_\mathbf{x}$ is non-zero in a single degree. The conclusion follows. □

The fact that the above lemma refers to the support of $\mathbf{P}^\bullet$, which is the closure of the set of points with non-zero stalk cohomology, means that we can use it to conclude something about the closure of the $\mathbf{P}^\bullet$-critical locus (recall Definition 1.9).

Theorem 3.2. *Let $\mathbf{P}^\bullet$ be a perverse sheaf on X, and suppose that the characteristic cycle of $\mathbf{P}^\bullet$ in $\mathcal{U}$ is given by $\mathrm{Ch}(\mathbf{P}^\bullet) = \sum_\alpha m_\alpha \left[\overline{T^*_{S_\alpha}\mathcal{U}}\right]$.*

Then, the closure of the $\mathbf{P}^\bullet$-critical locus of f is given by

$$\overline{\Sigma_{\mathbf{P}^\bullet} f} \;=\; \left\{\mathbf{x} \in X \;\middle|\; (\mathbf{x}, d_\mathbf{x}\tilde f) \in |\,\mathrm{Ch}(\mathbf{P}^\bullet)|\right\} \;=\; \bigcup_{m_\alpha \neq 0} \Sigma_{\rm cnr}\big(f_{|_{\overline{S}_\alpha}}\big).$$

Proof. Let $\mathbf{q} \in X$, and let $v = f(\mathbf{q})$. Let $\mathcal{W}$ be an open neighborhood of $\mathbf{q}$ in X such that $\mathcal{W} \cap \Sigma_{\mathbf{P}^\bullet} f \subseteq V(f-v)$ (see the end of Remark 1.7). Then, $\mathcal{W} \cap \overline{\Sigma_{\mathbf{P}^\bullet} f} = \mathcal{W} \cap \mathrm{supp}\,\phi_{f-v}\mathbf{P}^\bullet$. As $\phi_{f-v}\mathbf{P}^\bullet[-1]$ is perverse, Lemma 3.1 tells us that $\mathrm{supp}\,\phi_{f-v}\mathbf{P}^\bullet$ equals the image in X of $\mathrm{Ch}(\phi_{f-v}\mathbf{P}^\bullet)$. Now, Theorem 2.10 tells us that this image is precisely

$$\bigcup_{m_\alpha \neq 0} \left\{\mathbf{x} \in \overline{S_\alpha} \mid (\mathbf{x}, d_\mathbf{x}\tilde f) \in \overline{T^*_{S_\alpha}\mathcal{U}}\right\},$$

since there can be no cancellation as all the non-zero m_α have the same sign.

Therefore, we have the desired equality of sets in an open neighborhood of every point; the theorem follows. □

We will use the *perverse cohomology* of the shifted constant sheaf, $\mathbb{C}^\bullet_X[k]$, in order to deal with non-l.c.i.'s; this perverse cohomology is denoted by ${}^{p}H^0(\mathbb{C}^\bullet_X[k])$ (see [**BBD**] or [**K-S**]). Like the intersection cohomology complex, this sheaf has the property that it is the shifted constant sheaf on the smooth part of any component of X with dimension equal to $\dim X$.

We now list some properties of the perverse cohomology and of vanishing cycles that we will need later. The reader is referred to [**BBD**] and [**K-S**].

The perverse cohomology functor on X, ${}^{p}H^0$, is a functor from the derived category of bounded, constructible complexes on X to the Abelian category of perverse sheaves on X.

The functor ${}^{p}H^0$, applied to a perverse sheaf $\mathbf{P}^\bullet$ is canonically isomorphic to $\mathbf{P}^\bullet$. In addition, a bounded, constructible complex of sheaves $\mathbf{F}^\bullet$ is perverse if and only ${}^{p}H^0(\mathbf{F}^\bullet[k]) = 0$ for all $k \neq 0$. In particular, if X is an l.c.i., then ${}^{p}H^0(\mathbb{C}^\bullet_X[\dim X]) \cong \mathbb{C}^\bullet_X[\dim X]$ and ${}^{p}H^0(\mathbb{C}^\bullet_X[k]) = 0$ if $k \neq \dim X$.

The functor ${}^{p}H^0$ commutes with vanishing cycles with a shift of -1, nearby cycles with a shift of -1, and Verdier dualizing. That is, there are natural isomorphisms

$$ {}^{p}H^0 \circ \phi_f[-1] \cong \phi_f[-1] \circ {}^{p}H^0, \quad {}^{p}H^0 \circ \psi_f[-1] \cong \psi_f[-1] \circ {}^{p}H^0, \quad \text{and } \mathcal{D} \circ {}^{p}H^0 \cong {}^{p}H^0 \circ \mathcal{D}. $$

Let $\mathbf{F}^\bullet$ be a bounded complex of sheaves on X which is constructible with respect to a connected Whitney stratification $\{S_\alpha\}$ of X. Let $S_{\max}$ be a maximal stratum contained in the support of $\mathbf{F}^\bullet$, and let $m = \dim S_{\max}$. Then, $\left({}^{p}H^0(\mathbf{F}^\bullet)\right)_{|_{S_{\max}}}$ is isomorphic (in the derived category) to the complex which has $(\mathbf{H}^{-m}(\mathbf{F}^\bullet))_{|_{S_{\max}}}$ in degree $-m$ and zero in all other degrees.

In particular, $\operatorname{supp} \mathbf{F}^\bullet = \bigcup_i \operatorname{supp} {}^{p}H^0(\mathbf{F}^\bullet[i])$, and if $\mathbf{F}^\bullet$ is supported on an isolated point, $\mathbf{q}$, then $H^0({}^{p}H^0(\mathbf{F}^\bullet))_{\mathbf{q}} \cong H^0(\mathbf{F}^\bullet)_{\mathbf{q}}$.

Throughout the remainder of this paper, we let ${}^{k}\mathbf{P}^\bullet$ denote the perverse sheaf ${}^{p}H^0(\mathbb{C}^\bullet_X[k+1])$; it will be useful later to have a nice characterization of the characteristic cycle of ${}^{k}\mathbf{P}^\bullet$.

Proposition 3.3. *The complex* ${}^{k}\mathbf{P}^\bullet$ *is a perverse sheaf on* X *which is constructible with respect to* $\mathcal{S}$ *and the characteristic cycle* $\mathrm{Ch}({}^{k}\mathbf{P}^\bullet)$ *is equal to*

$$ (-1)^{\dim X} \sum_\alpha b_{k+1-d_\alpha}(N_\alpha, \mathbb{L}_\alpha) \left[\overline{T^*_{S_\alpha}\mathcal{U}}\right], $$

where b_j *denotes the* j*-th (relative) Betti number.*

Proof. The constructibility claim follows from the fact that the constant sheaf itself is clearly constructible with respect to any Whitney stratification. The remainder follows trivially from the definition of the characteristic cycle, combined with two properties of ${}^{p}H^0$; namely, ${}^{p}H^0$ commutes with $\phi_f[-1]$, and ${}^{p}H^0$ applied to a complex which is supported at a point simply gives ordinary cohomology in degree zero and zeroes in all other degrees. See [**K-S**, 10.3]. □

Remark 3.4. As N_α is contractible, it is possible to give a characterization of $b_{k+1-d_\alpha}(N_\alpha, \mathbb{L}_\alpha)$ without referring to N_α; the statement gets a little complicated, however, since we have to worry about what happens near degree zero and because the link of a maximal stratum is empty. However, if we slightly modify the usual definitions of reduced cohomology and the corresponding reduced Betti numbers, then the statement becomes quite easy.

What we want is for the "reduced" cohomology $\widetilde{H}^k(A;\mathbb{C})$ to be the relative cohomology vector space $H^{k+1}(B,A;\mathbb{C})$, where B is a contractible set containing A, and we want $\tilde{b}_*()$ to be the Betti numbers of this "reduced" cohomology. Therefore, letting $b_k()$ denote the usual k-th Betti number, we define $\tilde{b}_*()$ by

$$\tilde{b}_k(A) = \begin{cases} b_k(A), & \text{if } k \neq 0 \text{ and } A \neq \emptyset \\ b_0(A) - 1, & \text{if } k = 0 \text{ and } A \neq \emptyset \\ 0, & \text{if } k \neq -1 \text{ and } A = \emptyset \\ 1, & \text{if } k = -1 \text{ and } A = \emptyset. \end{cases}$$

Thus, $\tilde{b}_k(A)$ is the k-th Betti number of the reduced cohomology, provided that A is not the empty set.

We let $\widetilde{H}^k(A;\mathbb{C})$ denote the vector space $\mathbb{C}^{\tilde{b}_k(A)}$.

With this notation, the expression $b_{k+1-d_\alpha}(N_\alpha, \mathbb{L}_\alpha)$, which appears in 3.3, is equal to $\tilde{b}_{k-d_\alpha}(\mathbb{L}_\alpha)$. The special definition of $\tilde{b}_k()$ for the empty set implies that if S_α is maximal, then

$$b_{k+1-d_\alpha}(N_\alpha, \mathbb{L}_\alpha) = \begin{cases} 0, & \text{if } k+1 \neq d_\alpha \\ 1, & \text{if } k+1 = d_\alpha. \end{cases}$$

Hence, if $\operatorname{Ch}({}^k\mathbf{P}^\bullet) = \sum_\alpha m_\alpha\big({}^k\mathbf{P}^\bullet\big)\Big[\overline{T^*_{S_\alpha}\mathcal{U}}\Big]$, then 3.3 implies that: $H^*(\mathbb{L}_\alpha;\mathbb{C}) \cong H^*(point;\mathbb{C})$ if and only if $m_\alpha\big({}^k\mathbf{P}^\bullet\big) = 0$ for all k.

By combining 3.2 with 3.3 and 3.4, we can now give a result about $\Sigma_{\mathbb{C}} f$. First, though, it will be useful to adopt the following terminology.

Definition 3.5. We say that the stratum S_α is *visible* (or, $\mathbb{C}$-*visible*) if $H^*(\mathbb{L}_\alpha;\mathbb{C}) \ncong H^*(point;\mathbb{C})$ (or, equivalently, if $H^*(N_\alpha, \mathbb{L}_\alpha;\mathbb{C}) \neq 0$). Otherwise, the stratum is *invisible*.

The final line of Remark 3.4 tells us that a stratum is visible if and only if there exists an integer k such that $\Big[\overline{T^*_{S_\alpha}\mathcal{U}}\Big]$ appears with a non-zero coefficient in $\operatorname{Ch}\big({}^k\mathbf{P}^\bullet\big)$.

Note that if S_α has an empty complex link (i.e., the stratum is maximal), then S_α is **visible**.

Theorem 3.6. *Then,*

$$\overline{\Sigma_{\mathbb{C}} f} \;=\; \bigcup_{k=-1}^{\dim X - 1} \overline{\Sigma_{{}^k\mathbf{P}^\bullet} f} \;=\; \bigcup_{\text{visible } S_\alpha} \Big\{\mathbf{x} \in \overline{S_\alpha} \mid (\mathbf{x}, d_{\mathbf{x}}\tilde{f}) \in \overline{T^*_{S_\alpha}\mathcal{U}}\Big\} \;=\; \bigcup_{\text{visible } S_\alpha} \Sigma_{\mathrm{cnr}}\big(f_{|_{\overline{S}_\alpha}}\big).$$

In particular, since all maximal strata are visible, $\Sigma_{\mathrm{cnr}} f \subseteq \overline{\Sigma_{\mathbb{C}} f}$ (as stated in Proposition 1.6). Moreover, if $\mathbf{x}$ is an isolated point of $\Sigma_{\mathbb{C}} f$, then, for all Whitney stratifications, $\{R_\beta\}$, of X, the only possibly visible stratum which can be contained in $f^{-1}f(\mathbf{x})$ is $\{\mathbf{x}\}$.

Proof. Recall that, for any complex $\mathbf{F}^\bullet$, $\operatorname{supp}\mathbf{F}^\bullet = \bigcup_k \operatorname{supp}{}^p\!H^0(\mathbf{F}^\bullet[k])$. In addition, we claim that ${}^k\mathbf{P}^\bullet = 0$ unless $-1 \leqslant k \leqslant \dim X - 1$. By Lemma 3.1, ${}^k\mathbf{P}^\bullet = 0$ is equivalent to $\operatorname{Ch}({}^k\mathbf{P}^\bullet) = 0$; if k is not between -1 and $\dim X-1$, then, using Proposition 3.3, $\mathrm{Ch}({}^k\mathbf{P}^\bullet)=0$ follows from the fact that the complex link of a stratum has the homotopy-type of a finite CW complex of dimension no more than the complex dimension of the link (see [**Go-Mac1**]).

Now, in an open neighborhood of any point $\mathbf{q}$ with $v:=f(\mathbf{q})$, we have

$$\overline{\Sigma_{\mathbb{C}}f} \;=\; \operatorname{supp}\phi_{f-v}\mathbb{C}^\bullet \;=\; \bigcup_k \operatorname{supp}{}^pH^0(\phi_{f-v}\mathbb{C}^\bullet_X[k]) \;=$$

$$\bigcup_k \operatorname{supp}\phi_{f-v}[-1]\big({}^pH^0(\mathbb{C}^\bullet_X[k+1])\big) \;=\; \bigcup_k \overline{\Sigma_{{}^k\mathbf{P}^\bullet}f}.$$

Now, applying Theorem 3.2, we have

$$\overline{\Sigma_{\mathbb{C}}f} \;=\; \bigcup_k \bigcup_{m_\alpha({}^k\mathbf{P}^\bullet)\neq 0} \Big\{\mathbf{x}\in\overline{S_\alpha} \mid (\mathbf{x}, d_{\mathbf{x}}\tilde{f})\in\overline{T^*_{S_\alpha}\mathcal{U}}\Big\}.$$

The desired conclusion follows. □

Remark 3.7. Those familiar with stratified Morse theory should find the result of Theorem 3.6 very un-surprising – it looks like it results from some break-down of the $\mathbb{C}$-critical locus into normal and tangential data, and naturally one gets no contributions from strata with trivial normal data. This is the approach that we took in Theorem 3.2 of [**Ma1**]. There is a slightly subtle, technical point which prevents us from taking this approach in our current setting: by taking normal slices at points in an open, dense subset of $\operatorname{supp}\phi_{f-v}\mathbb{C}^\bullet_X$, we could reduce ourselves to the case where $\Sigma_{\mathbb{C}}f$ consists of a single point, but we would **not** know that the point was a **stratified** isolated critical point. In particular, the case where $\operatorname{supp}\phi_{f-v}\mathbb{C}^\bullet_X$ consists of a single point, but where f has a non-isolated (stratified) critical locus coming from an invisible stratum causes difficulties with the obvious Morse Theory approach.

Remark 3.8. At this point, we wish to add to our hierarchy of critical loci from Proposition 1.6. Theorem 3.6 tells us that $\overline{\Sigma_{{}^k\mathbf{P}^\bullet}f}\subseteq\overline{\Sigma_{\mathbb{C}}f}$ for all k. If X is purely $(d+1)$-dimensional, then 3.2 implies that $\Sigma_{\mathrm{cnr}}f\subseteq\overline{\Sigma_{{}^d\mathbf{P}^\bullet}f}$.

Now, suppose that X is irreducible of dimension $d+1$. Let $\mathbf{IC}^\bullet$ be the intersection cohomology sheaf (with constant coefficients) on X (see [**Go-Mac2**]); $\mathbf{IC}^\bullet$ is a simple object in the category of perverse sheaves. As the category of perverse sheaves on X is (locally) Artinian, and since ${}^d\mathbf{P}^\bullet$ is a perverse sheaf which is the shifted constant sheaf on the smooth part of X, it follows that $\mathbf{IC}^\bullet$ appears as a simple subquotient in any composition series for ${}^d\mathbf{P}^\bullet$. Consequently, $|\,\mathrm{Ch}(\mathbf{IC}^\bullet)|\subseteq|\,\mathrm{Ch}({}^d\mathbf{P}^\bullet)|$, and so 3.2 implies that $\overline{\Sigma_{\mathbf{IC}^\bullet}f}\subseteq\overline{\Sigma_{{}^d\mathbf{P}^\bullet}f}$. Moreover, 3.2 also implies that $\Sigma_{\mathrm{cnr}}f\subseteq\overline{\Sigma_{\mathbf{IC}^\bullet}f}$. Therefore, we can extend our sequence of inclusions from Proposition 1.6 to:

$$\overline{\Sigma_{\mathrm{reg}}f}\;\subseteq\;\overline{\Sigma_{\mathrm{alg}}f}\;\subseteq\;\overline{\Sigma_{\mathrm{Nash}}f}\;\subseteq\;\Sigma_{\mathrm{cnr}}f\;\subseteq\;\overline{\Sigma_{\mathbf{IC}^\bullet}f}\;\subseteq\;\overline{\Sigma_{{}^d\mathbf{P}^\bullet}f}\subseteq\;\overline{\Sigma_{\mathbb{C}}f}\;\subseteq\;\Sigma_{\mathrm{can}}f\;\subseteq\;\Sigma_{\mathrm{rdf}}f.$$

Why not use one of these new critical loci as our most fundamental notion of the critical locus of f? Both $\Sigma_{\mathbf{IC}^\bullet}f$ and $\Sigma_{{}^d\mathbf{P}^\bullet}f$ are topological in nature, and easy examples show that they can be distinct from $\Sigma_{\mathbb{C}}f$. However, 3.6 tells us that $\Sigma_{{}^d\mathbf{P}^\bullet}f$ is merely one piece that goes into making up $\Sigma_{\mathbb{C}}f$ – we should include the other shifted perverse cohomologies. On the other hand, given the importance of intersection cohomology throughout mathematics, one should wonder why we do not use $\Sigma_{\mathbf{IC}^\bullet}f$ as our most basic notion.

Consider the node $X:=V(y^2-x^3-x^2)\subseteq\mathbb{C}^2$ and the function $f:=y_{|_X}$. The node has a *small resolution of singularities* (see [**Go-Mac2**]) given by simply pulling the branches apart. As a result, the intersection cohomology sheaf on X is the constant sheaf shifted by one on $X-\{\mathbf{0}\}$, and the stalk cohomology at $\mathbf{0}$ is a copy of $\mathbb{C}^2$ concentrated in degree -1. Therefore, one can easily show that $\mathbf{0}\notin\overline{\Sigma_{\mathbf{IC}^\bullet}f}$.

As $\overline{\Sigma_{\mathbf{IC}^\bullet} f}$ fails to detect the simple change in topology of the level hypersurfaces of f as they go from being two points to being a single point, we do not wish to use $\Sigma_{\mathbf{IC}^\bullet} f$ as our basic type of critical locus. That is not to say that $\Sigma_{\mathbf{IC}^\bullet} f$ is not interesting in its own right; it is integrally tied to resolutions of singularities. For instance, it is easy to show (using the Decomposition Theorem [**BBD**]) that if $\widetilde{X} \xrightarrow{\pi} X$ is a resolution of singularities, then $\Sigma_{\mathbf{IC}^\bullet} f \subseteq \pi(\Sigma(f \circ \pi))$.

Now that we can "calculate" $\Sigma_{\mathbb{C}} f$ using Theorem 3.6, we are ready to generalize the Milnor number of a function with an isolated critical point.

Definition 3.9. If $\mathbf{P}^\bullet$ is a perverse sheaf on X, and $\mathbf{x}$ is an isolated point in $\Sigma_{\mathbf{P}^\bullet} f$ (or, if $\mathbf{x} \notin \Sigma_{\mathbf{P}^\bullet} f$), then we call $\dim_{\mathbb{C}} H^0(\phi_{f-f(\mathbf{x})}[-1]\mathbf{P}^\bullet)_{\mathbf{x}}$ the *Milnor number of f at $\mathbf{x}$ with coefficients in $\mathbf{P}^\bullet$* and we denote it by $\mu_{\mathbf{x}}(f; \mathbf{P}^\bullet)$.

This definition is reasonable for, in this case, $\phi_{f-f(\mathbf{x})}[-1]\mathbf{P}^\bullet$ is a perverse sheaf supported at the isolated point $\mathbf{x}$. Hence, the stalk cohomology of $\phi_{f-f(\mathbf{x})}[-1]\mathbf{P}^\bullet$ at $\mathbf{x}$ is possibly non-zero only in degree zero. Normally, we summarize that $\mathbf{x}$ is an isolated point in $\Sigma_{\mathbf{P}^\bullet} f$ or that $\mathbf{x} \notin \Sigma_{\mathbf{P}^\bullet} f$ by writing $\dim_{\mathbf{x}} \Sigma_{\mathbf{P}^\bullet} f \leqslant 0$ (we consider the dimension of the empty set to be $-\infty$).

Before we state the next proposition, note that it is always the case that $\left(\operatorname{im} d\tilde{f} \ \cdot \ T^*_{\{\mathbf{0}\}}\mathcal{U}\right)_{(\mathbf{0}, d_{\mathbf{0}}\tilde{f})} = 1$.

Proposition 3.10. *For notational convenience, we assume that $\mathbf{0} \in X$ and that $f(\mathbf{0}) = 0$.*
Then, $\dim_{\mathbf{0}} \Sigma_{\mathbb{C}} f \leqslant 0$ if and only if, for all k, $\dim_{\mathbf{0}} \Sigma_{{}^k\mathbf{P}^\bullet} f \leqslant 0$. Moreover, if $\dim_{\mathbf{0}} \Sigma_{\mathbb{C}} f \leqslant 0$, then,

i) for all visible strata, S_α, such that $\dim S_\alpha \geqslant 1$, the intersection of $\operatorname{im} d\tilde{f}$ and $\overline{T^*_{S_\alpha}\mathcal{U}}$ is, at most, 0-dimensional at $(\mathbf{0}, d_{\mathbf{0}}\tilde{f})$,
and

$$\left(\operatorname{im} d\tilde{f} \ \cdot \ \overline{T^*_{S_\alpha}\mathcal{U}}\right)_{(\mathbf{0}, d_{\mathbf{0}}\tilde{f})} \ = \ \left(\Gamma^1_{f,L}(S_\alpha) \cdot V(f)\right)_{\mathbf{0}} - \left(\Gamma^1_{f,L}(S_\alpha) \cdot V(L)\right)_{\mathbf{0}},$$

where L is a generic linear form, and

ii) for all k,

$$\mu_{\mathbf{0}}(f; {}^k\mathbf{P}^\bullet) \ = \ \tilde{b}_k(F_{f,\mathbf{0}}) \ = \ (-1)^{\dim X}\left(\operatorname{im} d\tilde{f} \ \cdot \ \operatorname{Ch}({}^k\mathbf{P}^\bullet)\right)_{(\mathbf{0}, d_{\mathbf{0}}\tilde{f})} \ =$$

$$\sum_{\text{visible } S_\alpha} \tilde{b}_{k-d_\alpha}(\mathbb{L}_\alpha) \left(\operatorname{im} d\tilde{f} \ \cdot \ \overline{T^*_{S_\alpha}\mathcal{U}}\right)_{(\mathbf{0}, d_{\mathbf{0}}\tilde{f})} =$$

$$\sum_{\substack{\text{visible } S_\alpha \\ S_\alpha \text{ not maximal}}} \tilde{b}_{k-d_\alpha}(\mathbb{L}_\alpha) \left(\operatorname{im} d\tilde{f} \ \cdot \ \overline{T^*_{S_\alpha}\mathcal{U}}\right)_{(\mathbf{0}, d_{\mathbf{0}}\tilde{f})} \ + \sum_{\substack{S_\alpha \text{ maximal} \\ \dim S_\alpha = k+1}} \left(\operatorname{im} d\tilde{f} \ \cdot \ \overline{T^*_{S_\alpha}\mathcal{U}}\right)_{(\mathbf{0}, d_{\mathbf{0}}\tilde{f})}.$$

Proof. It follows immediately from 3.6 that $\dim_{\mathbf{0}} \Sigma_{\mathbb{C}} f \leqslant 0$ if and only if, for all k, $\dim_{\mathbf{0}} \Sigma_{{}^k\mathbf{P}^\bullet} f \leqslant 0$.

i) follows immediately from Lemma 2.7 (combined with Remark 2.11).

It remains for us to prove ii). As in the proof of 3.6, we have

$$ {}^{p}H^0(\phi_f\mathbb{C}^\bullet_X[k]) \ = \ \phi_f[-1]\big({}^{p}H^0(\mathbb{C}^\bullet_X[k+1])\big) = \ \phi_f[-1]^k\mathbf{P}^\bullet. $$

It follows that

$$ \mu_{\mathbf{0}}(f;{}^k\mathbf{P}^\bullet) = \dim_{\mathbb{C}} H^0(\phi_f[-1]^k\mathbf{P}^\bullet)_{\mathbf{0}} = \dim_{\mathbb{C}} H^0\big({}^{p}H^0(\phi_f\mathbb{C}^\bullet_X[k])\big)_{\mathbf{0}} = \dim_{\mathbb{C}} H^0\big(\phi_f\mathbb{C}^\bullet_X[k]\big)_{\mathbf{0}}, $$

where the last equality is a result of the fact that $\mathbf{0}$ is an isolated point in the support of $\phi_f\mathbb{C}^\bullet_X[k]$ (for properties of ${}^{p}H^0$, see the beginning of Section 2). Therefore,

$$ \mu_{\mathbf{0}}(f;{}^k\mathbf{P}^\bullet) = \dim_{\mathbb{C}} H^0\big(\phi_f\mathbb{C}^\bullet_X[k]\big)_{\mathbf{0}} = \dim_{\mathbb{C}} H^k\big(\phi_f\mathbb{C}^\bullet_X\big)_{\mathbf{0}} = \dim \widetilde{H}^k(F_{f,\mathbf{0}};\ \mathbb{C}). $$

That we also have the equality

$$ \mu_{\mathbf{0}}(f;{}^k\mathbf{P}^\bullet) = (-1)^{\dim X}\big(\operatorname{im} d\tilde{f} \ \cdot \ \operatorname{Ch}({}^k\mathbf{P}^\bullet)\big)_{(\mathbf{0},d_{\mathbf{0}}\tilde{f})} $$

is precisely the content of Theorem 2.10, interpreted as in the last paragraph of Remark 2.11.

The remaining equalities in ii) follow from the description of $\operatorname{Ch}({}^k\mathbf{P}^\bullet)$ given in Proposition 3.3 and Remark 3.4. □

Remark 3.11. The formulas from 3.10 provide a topological/algebraic method for “calculating” the Betti numbers of the Milnor fibre for isolated critical points on arbitrary spaces. It should not be surprising that the data that one needs is not just the algebraic data – coming from the polar curves and intersection numbers – but also includes topological data about the underlying space: one has to know the Betti numbers of the complex links of strata.

Example 3.12. The most trivial, non-trivial case where one can apply 3.10 is the case where X is an irreducible local, complete intersection with an isolated singularity (that is, X is an irreducible i.c.i.s). Let us assume that $\mathbf{0} \in X$ is the only singular point of X and that f has an isolated $\mathbb{C}$-critical point at $\mathbf{0}$. Let d denote the dimension of X.

Let us write $\mathbb{L}_{X,\mathbf{0}}$ for the complex link of X at $\mathbf{0}$. By [**Lê1**], $\mathbb{L}_{X,\mathbf{0}}$ has the homotopy-type of a finite bouquet of $(d-1)$-spheres. Applying 3.10.ii, we see, then, that the reduced cohomology of $F_{f,\mathbf{0}}$ is concentrated in degree $(d-1)$, and the $(d-1)$-th Betti number of $F_{f,\mathbf{0}}$ is equal to

$$ \tilde{b}_{d-1}(\mathbb{L}_{X,\mathbf{0}})\Big(\operatorname{im} d\tilde{f} \ \cdot \ T^*_{\mathbf{0}}\mathcal{U}\Big)_{(\mathbf{0},d_{\mathbf{0}}\tilde{f})} \ + \ \Big(\operatorname{im} d\tilde{f} \ \cdot \ \overline{T^*_{X_{\mathrm{reg}}}\mathcal{U}}\Big)_{(\mathbf{0},d_{\mathbf{0}}\tilde{f})} = $$

$$ \tilde{b}_{d-1}(\mathbb{L}_{X,\mathbf{0}}) \ + \ \big(\Gamma^1_{f,L}(X_{\mathrm{reg}})\cdot V(f)\big)_{\mathbf{0}} - \big(\Gamma^1_{f,L}(X_{\mathrm{reg}})\cdot V(L)\big)_{\mathbf{0}}, $$

for generic linear L.

Now, the polar curve and the intersection numbers are quite calculable in practice; see Remark 1.8 and Example 1.9 of [**Ma1**]. However, there remains the question of how one can compute $\tilde{b}_{d-1}(\mathbb{L}_{X,\mathbf{0}})$. Corollary 4.6 and Example 5.4 of [**Ma1**] provide an inductive method for computing the **Euler characteristic** of $\mathbb{L}_{X,\mathbf{0}}$ (the induction is on the codimension of X in $\mathcal{U}$) and, since we know that $\mathbb{L}_{X,\mathbf{0}}$ has the homotopy-type of a bouquet of spheres, knowing the Euler characteristic is equivalent to knowing $\tilde{b}_{d-1}(\mathbb{L}_{X,\mathbf{0}})$.

The obstruction to using 3.10 to calculate Betti numbers in the general case is that, if X is not an l.c.i., then a formula for the Euler characteristic of the link of a stratum does not tell us the Betti numbers of the link.

Section 4. THOM'S a_f CONDITION.

We continue with the notation from Section 2.

In this section, we explain the fundamental relationship between Thom's a_f condition and the vanishing cycles of f.

Definition 4.1. Let M and N be analytic submanifolds of X such that f is constant on N. Then, the pair (M, N) *satisfies Thom's a_f condition at a point* $\mathbf{x} \in N$ if and only if we have the containment $\left(\overline{T^*_{f_{|_M}}\mathcal{U}}\right)_{\mathbf{x}} \subseteq \left(T^*_N\mathcal{U}\right)_{\mathbf{x}}$ of fibres over $\mathbf{x}$.

We have been slightly more general in the above definition than is sometimes the case; we have not required that the rank of f be constant on M. Thus, if X is an analytic space, we may write that (X_{reg}, N) satisfies the a_f condition, instead of writing the much more cumbersome $(X_{\mathrm{reg}} - \Sigma\big(f_{|_{X_{\mathrm{reg}}}}\big), N)$ satisfies the a_f condition. If f is not constant on any irreducible component of X, it is easy to see that these statements are equivalent:

Let $\overset{\circ}{X} := X_{\mathrm{reg}} - \Sigma\big(f_{|_{X_{\mathrm{reg}}}}\big)$, which is dense in X_{reg} (as f is not constant on any irreducible components of X). We claim that $\mathbb{P}\big(\overline{T^*_{f_{|_{\overset{\circ}{X}}}}\mathcal{U}}\big) = \mathbb{P}\big(\overline{T^*_{f_{|_{X_{\mathrm{reg}}}}}\mathcal{U}}\big)$; clearly, this is equivalent to showing that $T^*_{f_{|_{X_{\mathrm{reg}}}}}\mathcal{U} \subseteq \overline{T^*_{f_{|_{\overset{\circ}{X}}}}\mathcal{U}}$. This is simple, for if $\mathbf{x} \in \Sigma\big(f_{|_{X_{\mathrm{reg}}}}\big)$, then $(\mathbf{x}, \eta) \in T^*_{f_{|_{X_{\mathrm{reg}}}}}\mathcal{U}$ if and only if $(\mathbf{x}, \eta) \in T^*_{X_{\mathrm{reg}}}\mathcal{U}$, and $T^*_{X_{\mathrm{reg}}}\mathcal{U} \subseteq \overline{T^*_{\overset{\circ}{X}}\mathcal{U}} \subseteq \overline{T^*_{f_{|_{\overset{\circ}{X}}}}\mathcal{U}}$.

The link between Theorem 2.10 and the a_f condition is provided by the following theorem, which describes the fibre in the relative conormal in terms of the exceptional divisor in the blow-up of $\operatorname{im} d\tilde{f}$. Originally, we needed to assume Whitney's condition a) as an extra hypothesis; however, T. Gaffney showed us how to remove this assumption by using a re-parameterization trick.

Theorem 4.2. *Let $\pi : \mathcal{U} \times \mathbb{C}^{n+1} \times \mathbb{P}^n \to \mathcal{U} \times \mathbb{P}^n$ denote the projection.*

Suppose that f does not vanish identically on any irreducible component of X. Let E denote the exceptional divisor in $\mathrm{Bl}_{\operatorname{im} d\tilde{f}}\, \overline{T^*_{X_{\mathrm{reg}}}\mathcal{U}} \subseteq \mathcal{U} \times \mathbb{C}^{n+1} \times \mathbb{P}^n$.

Then, for all $\mathbf{x} \in X$, there is an inclusion of fibres over $\mathbf{x}$ given by $\big(\pi(E)\big)_{\mathbf{x}} \subseteq \left(\mathbb{P}\big(\overline{T^*_{f_{|_{X_{\mathrm{reg}}}}}\mathcal{U}}\big)\right)_{\mathbf{x}}$. Moreover, if $\mathbf{x} \in \Sigma_{\mathrm{Nash}} f$, then this inclusion is actually an equality.

Proof. By 2.12, it does not matter what extension of f we use.

That $\big(\pi(E)\big)_{\mathbf{x}} \subseteq \left(\mathbb{P}\big(\overline{T^*_{f_{|_{X_{\mathrm{reg}}}}}\mathcal{U}}\big)\right)_{\mathbf{x}}$ is easy. Suppose that $(\mathbf{x}, [\eta]) \in \pi(E)$, that is $(\mathbf{x}, d_{\mathbf{x}}\tilde{f}, [\eta]) \in E$. Then, there exists a sequence $(\mathbf{x}_i, \omega_i) \in \overline{T^*_{X_{\mathrm{reg}}}\mathcal{U}} - \operatorname{im} d\tilde{f}$ such that $(\mathbf{x}_i, \omega_i, [\omega_i - d_{\mathbf{x}_i}\tilde{f}]) \to (\mathbf{x}, d_{\mathbf{x}}\tilde{f}, [\eta])$. Hence, there exist scalars a_i such that $a_i(\omega_i - d_{\mathbf{x}_i}\tilde{f}) \to \eta$, and these $a_i(\omega_i - d_{\mathbf{x}_i}\tilde{f})$ are relative conormal covectors whose projective class approaches that of η. Thus, $\big(\pi(E)\big)_{\mathbf{x}} \subseteq \left(\mathbb{P}\big(\overline{T^*_{f_{|_{X_{\mathrm{reg}}}}}\mathcal{U}}\big)\right)_{\mathbf{x}}$.

We must now show that $\left(\mathbb{P}\big(\overline{T^*_{f_{|_{X_{\mathrm{reg}}}}}\mathcal{U}}\big)\right)_{\mathbf{x}} \subseteq \big(\pi(E)\big)_{\mathbf{x}}$, provided that $\mathbf{x} \in \Sigma_{\mathrm{Nash}} f$.

Let $\overset{\circ}{X} := X_{\rm reg} - \Sigma\big(f_{|_{X_{\rm reg}}}\big)$. Suppose that $(\mathbf{x}, [\eta]) \in \mathbb{P}\big(\overline{T^*_{f_{|_{\overset{\circ}{X}}}}\mathcal{U}}\big)$. Then, there exists a complex analytic path $\alpha(t) = (\mathbf{x}(t), \eta_t) \in \overline{T^*_{f_{|_{\overset{\circ}{X}}}}\mathcal{U}}$ such that $\alpha(0) = (\mathbf{x}, \eta)$ and $\alpha(t) \in T^*_{f_{|_{\overset{\circ}{X}}}}\mathcal{U}$ for $t \neq 0$. As f has no critical points on $\overset{\circ}{X}$, each η_t can be written uniquely as $\eta_t = \omega_t + \lambda(\mathbf{x}(t)) d_{\mathbf{x}(t)}\tilde{f}$, where $\omega_t(T_{\mathbf{x}(t)}\overset{\circ}{X}) = 0$ and $\lambda(\mathbf{x}(t))$ is a scalar. By evaluating each side on $\mathbf{x}'(t)$, we find that $\lambda(\mathbf{x}(t)) = \frac{\eta_t(\mathbf{x}'(t))}{\frac{d}{dt}f(\mathbf{x}(t))}$.

Thus, as $\lambda(\mathbf{x}(t))$ is a quotient of two analytic functions, there are only two possibilities for what happens to $\lambda(\mathbf{x}(t))$ as $t \to 0$.

Case 1: $|\lambda(\mathbf{x}(t))| \to \infty$ as $t \to 0$.

In this case, since $\eta_t \to \eta$, it follows that $\dfrac{\eta_t}{\lambda(\mathbf{x}(t))} \to 0$ and, hence, $-\dfrac{\omega_t}{\lambda(\mathbf{x}(t))} \to d_{\mathbf{x}}\tilde{f}$. Therefore,

$$\left(\mathbf{x}(t), -\frac{\omega_t}{\lambda(\mathbf{x}(t))}, \left[-\frac{\omega_t}{\lambda(\mathbf{x}(t))} - d_{\mathbf{x}(t)}\tilde{f}\right]\right) = \left(\mathbf{x}(t), -\frac{\omega_t}{\lambda(\mathbf{x}(t))}, [\eta_t(\mathbf{x}(t))]\right) \to (\mathbf{x}, d_{\mathbf{x}}\tilde{f}, [\eta]),$$

and so $(\mathbf{x}, [\eta]) \in \pi(E)$.

Case 2: $\lambda(\mathbf{x}(t)) \to \lambda_0$ as $t \to 0$.

In this case, ω_t must possess a limit as $t \to 0$. For t small and unequal to zero, let proj_t denote the complex orthogonal projection from the fibre $\big(T^*_{f_{|_{\overset{\circ}{X}}}}\mathcal{U}\big)_{\mathbf{x}(t)}$ to the fibre $\big(T^*_{\overset{\circ}{X}}\mathcal{U}\big)_{\mathbf{x}(t)}$. Let $\gamma_t := \mathrm{proj}_t(\eta_t) = \omega_t + \lambda(\mathbf{x}(t))\,\mathrm{proj}_t(d_{\mathbf{x}(t)}\tilde{f})$. Since $\mathbf{x} \in \Sigma_{\rm Nash} f$, we have that $\mathrm{proj}_t(d_{\mathbf{x}(t)}\tilde{f}) \to d_{\mathbf{x}}\tilde{f}$ and, thus, $\gamma_t \to \eta$.

As η is not zero (since it represents a projective class), we may define the (real, non-negative) scalar

$$a_t := \sqrt{\frac{||\,\mathrm{proj}_t(d_{\mathbf{x}(t)}\tilde{f}) - d_{\mathbf{x}(t)}\tilde{f}||}{||\gamma_t||}}.$$

One now verifies easily that

$$(\mathbf{x}(t),\ a_t\gamma_t + \mathrm{proj}_t(d_{\mathbf{x}(t)}\tilde{f}),\ [a_t\gamma_t + \mathrm{proj}_t(d_{\mathbf{x}(t)}\tilde{f}) - d_{\mathbf{x}(t)}\tilde{f}]) \longrightarrow (\mathbf{x}, d_{\mathbf{x}}\tilde{f}, [\eta]),$$

and, hence, that $(\mathbf{x}, [\eta]) \in \pi(E)$. □

Remark 4.3. In a number of results throughout the remainder of this paper, the reader will find the hypotheses that $\mathbf{x} \in \Sigma_{\rm Nash} f$ or that $\mathbf{x} \in \Sigma_{\rm alg} f$. While Theorem 4.2 explains why the hypothesis $\mathbf{x} \in \Sigma_{\rm Nash} f$ is important, it is not so clear why the hypothesis $\mathbf{x} \in \Sigma_{\rm alg} f$ is of interest.

If Y is an analytic subset of X, then one shows easily that $Y \cap \Sigma_{\rm alg} f \subseteq \Sigma_{\rm alg}(f_{|_Y})$. The Nash critical does not possess such an inheritance property. Thus, the easiest hypothesis to make in order to guarantee that a point, $\mathbf{x}$, is in the Nash critical locus of any analytic subset containing $\mathbf{x}$ is the hypothesis that $\mathbf{x} \in \Sigma_{\rm alg} f$, for then if $\mathbf{x} \in Y$, we conclude that $\mathbf{x} \in \Sigma_{\rm alg}(f_{|_Y}) \subseteq \Sigma_{\rm Nash}(f_{|_Y})$.

We come now to the result which tells one how the topological information provided by the sheaf of vanishing cycles controls the a_f condition.

Corollary 4.4. *Let N be a submanifold of X such that $N \subseteq V(f)$, and let $\mathbf{x} \in N$*

Let $\operatorname{Ch}(\mathbf{F}^\bullet) = \sum_\alpha m_\alpha \big[\overline{T^*_{M_\alpha}\mathcal{U}}\big]$, where $\{M_\alpha\}$ is a collection of connected analytic submanifolds of X such that either $m_\alpha \geqslant 0$ for all α, or $m_\alpha \leqslant 0$ for all α. Let $\operatorname{Ch}(\phi_f\mathbf{F}^\bullet) = \sum_\beta k_\beta \big[\overline{T^*_{R_\beta}\mathcal{U}}\big]$, where $\{R_\beta\}$ is a collection of connected analytic submanifolds.

Finally, suppose that, for all β, there is an inclusion of fibres over $\mathbf{x}$ given by $\Big(\overline{T^*_{R_\beta}\mathcal{U}}\Big)_{\mathbf{x}} \subseteq \big(T^*_N\mathcal{U}\big)_{\mathbf{x}}$.

Then, the pair $\Big(\big(\overline{M_\alpha}\big)_{\mathrm{reg}}, N\Big)$ satisfies Thom's a_f condition at $\mathbf{x}$ for every M_α for which $f_{|_{M_\alpha}} \not\equiv 0$, $m_\alpha \neq 0$ and such that $\mathbf{x} \in \Sigma_{\mathrm{Nash}}(f_{|_{\overline{M_\alpha}}})$.

Proof. Let $\{S_\gamma\}$ be a Whitney stratification for X such that each $\overline{M_\alpha}$ is a union of strata and such that $\Sigma\big(f_{|_{S_\gamma}}\big) = \emptyset$ unless $S_\gamma \subseteq V(f)$. Hence, for each α, there exists a unique S_γ such that $\overline{M_\alpha} = \overline{S_\gamma}$; denote this stratum by S_α. It follows at once that $\operatorname{Ch}(\mathbf{F}^\bullet) = \sum_\alpha m_\alpha \big[\overline{T^*_{S_\alpha}\mathcal{U}}\big]$.

From Theorem 2.10, $E = \sum_\alpha m_\alpha E_\alpha \cong \mathbb{P}\big(\operatorname{Ch}(\phi_f\mathbf{F}^\bullet)\big)$. Thus, since all non-zero m_α have the same sign, if m_α is not zero, then E_α appears with a non-zero coefficient in $\mathbb{P}\big(\operatorname{Ch}(\phi_f\mathbf{F}^\bullet)\big)$.

The result now follows immediately by applying Theorem 4.2 to each $\overline{M_\alpha}$ in place of X. $\square$

Theorem 4.2 also allows us to prove an interesting relationship between the characteristic varieties of the vanishing and nearby cycles – provided that the complex of sheaves under consideration is perverse.

Corollary 4.5. *Let $\mathbf{P}^\bullet$ be a perverse sheaf on X. If $\mathbf{x} \in \Sigma_{\mathrm{alg}} f$ and $(\mathbf{x}, \eta) \in |\operatorname{Ch}(\psi_f\mathbf{P}^\bullet)|$, then $(\mathbf{x}, \eta) \in |\operatorname{Ch}(\phi_f\mathbf{P}^\bullet)|$.*

Proof. Let $\mathcal{S} := \{S_\alpha\}$ be a Whitney stratification with connected strata such that $\mathbf{P}^\bullet$ is constructible with respect to $\mathcal{S}$ and such that $V(f)$ is a union of strata. For the remainder of the proof, we will work in a neighborhood of $V(f)$ – a neighborhood in which, if $S_\alpha \not\subseteq V(f)$, then $\Sigma(f_{|_{S_\alpha}}) = \emptyset$.

Let $\operatorname{Ch}(\mathbf{P}^\bullet) = \sum m_\alpha \Big[\overline{T^*_{S_\alpha}\mathcal{U}}\Big]$. As $\mathbf{P}^\bullet$ is perverse, all non-zero m_α have the same sign. Thus, 2.10 tells us – using the notation from 2.10 – that

$$(\dagger) \qquad |\mathbb{P}(\operatorname{Ch}(\phi_f\mathbf{P}^\bullet))| = \bigcup_{m_\alpha \neq 0} \pi(E_\alpha),$$

where E_α denotes the exceptional divisor in the blow-up of $\overline{T^*_{S_\alpha}\mathcal{U}}$ along $\operatorname{im} d\tilde{f}$ (in a neighborhood of $V(f)$). In addition, 2.3 tells us that

$$|\operatorname{Ch}(\psi_f\mathbf{P}^\bullet)| = \big(V(f) \times \mathbb{C}^{n+1}\big) \cap \bigcup_{\substack{m_\alpha \neq 0 \\ S_\alpha \not\subseteq V(f)}} \overline{T^*_{f_{|_{S_\alpha}}}\mathcal{U}}.$$

Assume $(\mathbf{x}, \eta) \in |\operatorname{Ch}(\psi_f\mathbf{P}^\bullet)|$. Then, there exists $S_\alpha \not\subseteq V(f)$ such that $m_\alpha \neq 0$ and $(\mathbf{x}, \eta) \in \overline{T^*_{f_{|_{S_\alpha}}}\mathcal{U}}$. Clearly, then, $(\mathbf{x}, \eta) \in \overline{T^*_{f_{|_{(\overline{S}_\alpha)_{\mathrm{reg}}}}}\mathcal{U}}$. Now, if $\mathbf{x} \in \Sigma_{\mathrm{alg}} f$ and $\eta \neq 0$, then $\mathbf{x} \in \Sigma_{\mathrm{alg}}(f_{|_{\overline{S}_\alpha}})$ and so Theorem 4.2 implies that $(\mathbf{x}, [\eta]) \in \pi(E_\alpha)$, where $[\eta]$ denotes the projective class of η and E_α denotes the exceptional divisor of the blow-up of $\overline{T^*_{(\overline{S}_\alpha)_{\mathrm{reg}}}\mathcal{U}} = \overline{T^*_{S_\alpha}\mathcal{U}}$ along $\operatorname{im} d\tilde{f}$. Thus, by $(\dagger)$, $(\mathbf{x}, \eta) \in |\operatorname{Ch}(\phi_f\mathbf{P}^\bullet)|$.

We are left with the trivial case of when $(\mathbf{x}, 0) \in |\operatorname{Ch}(\psi_f\mathbf{P}^\bullet)|$. Note that, if $(\mathbf{x}, 0) \in |\operatorname{Ch}(\psi_f\mathbf{P}^\bullet)|$, then there must exist some non-zero η such that $(\mathbf{x}, \eta) \in |\operatorname{Ch}(\psi_f\mathbf{P}^\bullet)|$. For, otherwise, the stratum (in some Whitney stratification) of $\operatorname{supp} \psi_f\mathbf{P}^\bullet$ containing $\mathbf{x}$ must be all of $\mathcal{U}$. However, $\psi_f\mathbf{P}^\bullet$ is supported on $V(f)$, and so f would have to be zero on all of $\mathcal{U}$; but, this implies that $|\operatorname{Ch}(\psi_f \mathbf{P}^\bullet)| = \emptyset$. Now, if we have some non-zero η such that $(\mathbf{x}, \eta) \in |\operatorname{Ch}(\psi_f \mathbf{P}^\bullet)|$, then by the above argument, $(\mathbf{x}, \eta) \in |\operatorname{Ch}(\phi_f \mathbf{P}^\bullet)|$ and, thus, certainly $(\mathbf{x}, 0) \in |\operatorname{Ch}(\phi_f \mathbf{P}^\bullet)|$. □

Section 5. CONTINUOUS FAMILIES OF CONSTRUCTIBLE COMPLEXES.

We wish to prove statements of the form: the constancy of certain data in a family implies that some nice geometric facts hold. As the reader should have gathered from the last section, it is very advantageous to use complexes of sheaves for cohomology coefficients; in particular, being able to use perverse coefficients is very desirable. The question arises: what should a family of complexes mean?

Let X be an analytic space, let $t : X \to \mathbb{C}$ be an analytic function, and let $\mathbf{F}^\bullet$ be a bounded, constructible complex of $\mathbb{C}$-vector spaces. We could say that $\mathbf{F}^\bullet$ and t form a "nice" family of complexes, since, for all $a \in \mathbb{C}$, we can consider the complex $\mathbf{F}^\bullet_{|_{t^{-1}(a)}}$ on the space $X_{|_{t^{-1}(a)}}$. However, this does yield a satisfactory theory, because there may be absolutely no relation between $\mathbf{F}^\bullet_{|_{t^{-1}(0)}}$ and $\mathbf{F}^\bullet_{|_{t^{-1}(a)}}$ for a close to 0. What we need is a notion of *continuous* families of complexes – we want $\mathbf{F}^\bullet_{|_{t^{-1}(0)}}$ to equal the "limit" of $\mathbf{F}^\bullet_{|_{t^{-1}(a)}}$ as a approaches 0. Fortunately, such a notion already exists; it just is not normally thought of as continuity.

Definition 5.1. Let X, t, and $\mathbf{F}^\bullet$ be as above. We define the *limit of* $\mathbf{F}^\bullet_a := \mathbf{F}^\bullet_{|_{t^{-1}(a)}}[-1]$ *as* a *approaches* b, $\lim\limits_{a \to b} \mathbf{F}^\bullet_a$, to be the nearby cycles $\psi_{t-b}\mathbf{F}^\bullet[-1]$.

We say that the family $\mathbf{F}^\bullet_a$ is *continuous at the value* b if the comparison map from $\mathbf{F}^\bullet_b$ to $\psi_{t-b}\mathbf{F}^\bullet[-1]$ is an isomorphism, i.e., if the vanishing cycles $\phi_{t-b}\mathbf{F}^\bullet[-1] = 0$. We say that the family $\mathbf{F}^\bullet_a$ is *continuous* if it is continuous for all values b.

We say that the family $\mathbf{F}^\bullet_a$ is *continuous at the point* $\mathbf{x} \in X$ if there is an open neighborhood $\mathcal{W}$ of $\mathbf{x}$ such that the family defined by restricting $\mathbf{F}^\bullet$ to $\mathcal{W}$ is continuous at the value $t(\mathbf{x})$.

If $\mathbf{P}^\bullet$ is a perverse sheaf on X and $\mathbf{P}^\bullet_a := \mathbf{P}^\bullet_{|_{t^{-1}(a)}}[-1]$ is a continuous family of complexes, then we say that $\mathbf{P}^\bullet_a$ is a continuous family of perverse sheaves.

Remark 5.2. The reason for the shifts by -1 in the families is so that if $\mathbf{P}^\bullet$ is perverse, and $\mathbf{P}^\bullet_a$ is a continuous family, then each $\mathbf{P}^\bullet_a$ is, in fact, a perverse sheaf (since $\mathbf{P}^\bullet_a \cong \psi_{t-a}\mathbf{P}^\bullet[-1]$).

It is not difficult to show that: if the family $\mathbf{F}^\bullet_a$ is continuous at the value b, and, for all $a \neq b$, each $\mathbf{F}^\bullet_a$ is perverse, then, near the vlaue b, the family $\mathbf{F}^\bullet_a$ is a continuous family of perverse sheaves.

For the remainder of this section, we will be using the following additional notation. Let $\tilde{t}$ be an analytic function on $\mathcal{U}$, and let t denote its restriction to X. Let $\mathbf{P}^\bullet$ be a perverse sheaf on X. Consider the families of spaces, functions, and sheaves given by $X_a := X \cap V(t-a)$, $f_a := f_{|_{X_a}}$, and $\mathbf{P}^\bullet_a := \mathbf{P}^\bullet_{|_{X_a}}[-1]$ (normally, if we are not looking at a specific value for t, we write X_t, f_t, and $\mathbf{P}^\bullet_t$ for these families). Note that, if we have as an hypothesis that $\mathbf{P}^\bullet_t$ is continuous, then the family $\mathbf{P}^\bullet_t$ is actually a family of **perverse** sheaves.

We will now prove three fundamental lemmas; all of them have trivial proofs, but they are nonetheless extremely useful.

The first lemma uses Theorem 3.2 to characterize continuity at a point for families of perverse sheaves.

Lemma 5.3. *Let $\mathbf{x} \in X$. The following are equivalent:*

i) *The family $\mathbf{P}^\bullet_t$ is continuous at $\mathbf{x}$;*

ii) $\mathbf{x} \notin \overline{\Sigma_{\mathbf{P}^\bullet} t}$;

iii) $(\mathbf{x}, d_{\mathbf{x}}\tilde{t}) \notin |\operatorname{Ch}(\mathbf{P}^\bullet)|$ *for some local extension, $\tilde{t}$, of t to $\mathcal{U}$ in a neighborhood of $\mathbf{x}$; and*

iv) $(\mathbf{x}, d_{\mathbf{x}}\tilde{t}) \notin |\operatorname{Ch}(\mathbf{P}^\bullet)|$ *for every local extension, $\tilde{t}$, of t to $\mathcal{U}$ in a neighborhood of $\mathbf{x}$.*

Proof. The equivalence of i) and ii) follows from their definitions, together with Remark 1.7. The equivalence between ii), iii), and iv) follows immediately from Theorem 3.2. □

The next lemma is a necessary step in several proofs.

Lemma 5.4. *Suppose that the family $\mathbf{P}^\bullet_t$ is continuous at $t = b$, and that the characteristic cycle of $\mathbf{P}^\bullet$ is given by $\sum_\alpha m_\alpha \left[\overline{T^*_{S_\alpha}\mathcal{U}}\right]$. Then, $S_\alpha \not\subseteq V(t-b)$ if $m_\alpha \neq 0$.*

Proof. This follows immediately from 5.3. □

The last of our three lemmas is the **stability of continuity** result.

Lemma 5.5. *Suppose that the family $\mathbf{P}^\bullet_t$ is continuous at $\mathbf{x} \in X$. Then, $\mathbf{P}^\bullet_t$ is continuous at all points near $\mathbf{x}$. In addition, if $\overset{\circ}{\mathbb{D}}$ is an open disk around the origin in $\mathbb{C}$, $h : \overset{\circ}{\mathbb{D}} \times X \to \mathbb{C}$ is an analytic function, $h_c(\mathbf{z}) := h(c, \mathbf{z})$, and $h_0 = t$, then the family $\mathbf{P}^\bullet_{h_c}$ is continuous at $\mathbf{x}$ for all c sufficiently close to 0.*

Proof. Let $\tilde{t}$ be an extension of t to a neighborhood of $\mathbf{x}$ in $\mathcal{U}$, and let $\Pi_1 : T^*\mathcal{U} \to \mathcal{U}$ be the cotangent bundle. As $T^*\mathcal{U}$ is isomorphic to $\mathcal{U} \times \mathbb{C}^{n+1}$, there is a second projection $\Pi_2 : T^*\mathcal{U} \to \mathbb{C}^{n+1}$.

Now, $\Pi_1^{-1}(\mathbf{x}) \cap |\operatorname{Ch}(\mathbf{P}^\bullet)|$ and $\Pi_2^{-1}(d_{\mathbf{x}}\tilde{t}) \cap |\operatorname{Ch}(\mathbf{P}^\bullet)|$ are closed sets. Therefore, the lemma follows immediately from 5.3. □

The following lemma allows us to use intersection-theoretic arguments for families of generalized isolated critical points.

Lemma 5.6. *Suppose that the family $\mathbf{P}^\bullet_t$ is continuous at $\mathbf{x} \in X$. Let $b := t(\mathbf{x})$. Let $\{S_\alpha\}$ be a Whitney stratification of X with connected strata with respect to which $\mathbf{P}^\bullet$ is constructible and such that $V(t-b)$ is a union of strata. Suppose that $\operatorname{Ch}(\mathbf{P}^\bullet)$ is given by $\sum_\alpha m_\alpha \left[\overline{T^*_{S_\alpha}\mathcal{U}}\right]$. If $\dim_{\mathbf{x}} \Sigma_{\mathbf{P}^\bullet_b} f_b \leqslant 0$, then there exists an open neighborhood $\mathcal{W}$ of $\mathbf{x}$ in $\mathcal{U}$ such that:*

i) $\operatorname{im} d\tilde{f}$ *properly intersects* $\displaystyle\sum_\alpha m_\alpha \left[\overline{T^*_{t_{|S_\alpha}}\mathcal{U}}\right]$ *in* $\mathcal{W}$;

ii) for all $\mathbf{y} \in X \cap \mathcal{W}$, $V(t - t(\mathbf{y}))$ *properly intersects*

$$\operatorname{im} d\tilde{f} \ \cdot \ \sum_{\alpha} m_\alpha \Big[\overline{T^*_{t_{|S_\alpha}}\mathcal{U}}\Big]$$

at $(\mathbf{y}, d_{\mathbf{y}}\tilde{f})$ *in (at most) an isolated point; and*

iii) for all $\mathbf{y} \in X \cap \mathcal{W}$, *if* $a := t(\mathbf{y})$, *then* $\dim_{\mathbf{y}} \Sigma_{\mathbf{P}^\bullet_a} f_a \leqslant 0$ *and*

$$\mu_{\mathbf{y}}(f_a; \mathbf{P}^\bullet_a) = (-1)^{\dim X} \Big[\Big(\operatorname{im} d\tilde{f} \ \cdot \ \sum_{\alpha} m_\alpha \Big[\overline{T^*_{t_{|S_\alpha}}\mathcal{U}}\Big]\Big) \ \cdot \ V(t-a)\Big]_{(\mathbf{y}, d_{\mathbf{y}}\tilde{f})}.$$

Proof. First, note that we may assume that $X = \operatorname{supp} \mathbf{P}^\bullet$; for, otherwise, we would immediately replace X by $\operatorname{supp} \mathbf{P}^\bullet$. Now, it follows from Lemma 5.4 that $V(t-b)$ does not contain an entire irreducible component of X. Thus $\dim X_0 = \dim X - 1$.

We use $\tilde{f}$ as a common extension of f_t to $\mathcal{U}$, for all t. Proposition 3.10 tells us that $\mu_{\mathbf{x}}(f_b; \mathbf{P}^\bullet_b) = (-1)^{\dim X - 1}\big(\operatorname{im} d\tilde{f} \ \cdot \ \operatorname{Ch}(\mathbf{P}^\bullet_b)\big)_{(\mathbf{b}, d_{\mathbf{b}}\tilde{f})}$. Then, continuity, implies that $\operatorname{Ch}(\mathbf{P}^\bullet_b) = \operatorname{Ch}(\psi_{t-b}[-1]\mathbf{P}^\bullet)$, and

$$(*) \qquad \operatorname{Ch}(\psi_{t-b}[-1]\mathbf{P}^\bullet) = -\operatorname{Ch}(\psi_{t-b}\mathbf{P}^\bullet) = -\big(V(t-b) \times \mathbb{C}^{n+1}\big) \ \cdot \ \sum_{S_\alpha \not\subseteq V(t-b)} m_\alpha \Big[\overline{T^*_{t_{|S_\alpha}}\mathcal{U}}\Big],$$

by Theorem 2.3. By Lemma 5.4, we may index over all S_α; for, if $S_\alpha \subseteq V(t-b)$, then $m_\alpha = 0$.

Therefore,

$$(\dagger) \qquad \mu_{\mathbf{x}}(f_b; \mathbf{P}^\bullet_b) = (-1)^{\dim X}\Big(\operatorname{im} d\tilde{f} \ \cdot \ \big(V(t-b) \times \mathbb{C}^{n+1}\big) \ \cdot \ \sum_{\alpha} m_\alpha \Big[\overline{T^*_{t_{|S_\alpha}}\mathcal{U}}\Big]\Big)_{(\mathbf{x}, d_{\mathbf{x}}\tilde{f})} =$$

$$(-1)^{\dim X}\Big(\Big(\operatorname{im} d\tilde{f} \ \cdot \ \sum_{\alpha} m_\alpha \Big[\overline{T^*_{t_{|S_\alpha}}\mathcal{U}}\Big]\Big) \ \cdot \ \big(V(t-b) \times \mathbb{C}^{n+1}\big)\Big)_{(\mathbf{x}, d_{\mathbf{x}}\tilde{f})}.$$

Thus,

$$C := (-1)^{\dim X}\Big(\operatorname{im} d\tilde{f} \ \cdot \ \sum_{\alpha} m_\alpha \Big[\overline{T^*_{t_{|S_\alpha}}\mathcal{U}}\Big]\Big)$$

is a non-negative cycle such that $(\mathbf{x}, d_{\mathbf{x}}\tilde{f})$ is an isolated point in (or, is not in) $C \cdot V(t-b)$. Statements i) and ii) of the lemma follow immediately.

Now, Lemma 5.5 tells us that the family $\mathbf{P}^\bullet_t$ is continuous at all points near $\mathbf{x}$; therefore, if $\mathbf{y}$ is close to $\mathbf{x}$ and $a := t(\mathbf{y})$, then, by repeating the argument for $(*)$, we find that

$$\operatorname{Ch}(\mathbf{P}^\bullet_a) = -\operatorname{Ch}(\psi_{t-a}\mathbf{P}^\bullet) = -\big(V(t-a) \times \mathbb{C}^{n+1}\big) \ \cdot \ \sum_{\alpha} m_\alpha \Big[\overline{T^*_{t_{|S_\alpha}}\mathcal{U}}\Big]$$

and we know that the intersection of this cycle with $\operatorname{im} d\tilde{f}$ is (at most) zero-dimensional at $(\mathbf{y}, d_{\mathbf{y}}\tilde{f})$ (since $C \cap V(t-b)$ is (at most) zero-dimensional at $\mathbf{x}$). By considering $\tilde{f}$ an extension of f_a and applying Theorem 3.2, we conclude that $\dim_{\mathbf{y}} \Sigma_{\mathbf{P}^\bullet_a} f_a \leqslant 0$.

Finally, now that we know that $\mathbf{P}^\bullet_t$ is continuous at $\mathbf{y}$ and that $\dim_{\mathbf{y}} \Sigma_{\mathbf{P}^\bullet_a} f_a \leqslant 0$, we may argue as we did at $\mathbf{x}$ to conclude that $(\dagger)$ holds with $\mathbf{x}$ replaced by $\mathbf{y}$ and b replaced by a. This proves iii). □

We can now prove an **additivity/upper-semicontinuity** result. We prove this result for a more general type of family of perverse sheaves; instead of parametrizing by the values of a function, we parametrize implicitly. We will need this more general perspective in Theorem 5.10.

Theorem 5.7. *Suppose that the family $\mathbf{P}^\bullet_t$ is continuous at $\mathbf{x} \in X$. Let $b := t(\mathbf{x})$, and suppose that* $\dim_{\mathbf{x}} \Sigma_{\mathbf{P}^\bullet_b} f_b \leqslant 0$.

Let $\overset{\circ}{\mathbb{D}}$ be an open disk around the origin in $\mathbb{C}$, let $h : \overset{\circ}{\mathbb{D}} \times X \to \mathbb{C}$ be an analytic function, for all $c \in \overset{\circ}{\mathbb{D}}$, let $h_c(\mathbf{z}) := h(c, \mathbf{z})$, let ${}_c\mathbf{P}^\bullet := \mathbf{P}^\bullet_{|_{V(h_c - b)}}[-1]$ and ${}_cf := f_{|_{V(h_c-b)}}$. Suppose that $h_0 = t$.

Then, there exists an open neighborhood $\mathcal{W}$ of $\mathbf{x}$ in $\mathcal{U}$ such that, for all small c, for all $\mathbf{y} \in V(h_c - b) \cap \mathcal{W}$, $\dim_{\mathbf{y}} \Sigma_{{}_c\mathbf{P}^\bullet}\, {}_cf \leqslant 0$.

Moreover, for fixed c close to 0, there are a finite number of points $\mathbf{y} \in V(h_c - b) \cap \mathcal{W}$ such that $\mu_{\mathbf{y}}({}_cf; {}_c\mathbf{P}^\bullet) \neq 0$ and

$$\mu_{\mathbf{x}}({}_bf; {}_b\mathbf{P}^\bullet) = \sum_{\mathbf{y} \in V(h_c - b) \cap \mathcal{W}} \mu_{\mathbf{y}}({}_cf; {}_c\mathbf{P}^\bullet).$$

In particular, for all small c, for all $\mathbf{y} \in V(h_c - b) \cap \mathcal{W}$, $\mu_{\mathbf{y}}({}_cf; {}_c\mathbf{P}^\bullet) \leqslant \mu_{\mathbf{x}}({}_bf; {}_b\mathbf{P}^\bullet)$.

Proof. We continue to let $\mathbf{P}^\bullet_c = \mathbf{P}^\bullet_{|_{V(t-c)}}[-1]$ and $f_c = f_{|_{V(t-c)}}$. Note that, if we let $h(w, \mathbf{z}) := t(\mathbf{z}) - w$, then the statement of the theorem would reduce to a statement about the ordinary families $\mathbf{P}^\bullet_c$ and f_c. Moreover, this statement about the families $\mathbf{P}^\bullet_c$ and f_c follows immediately from Lemma 5.6. We wish to see that this apparently weak form of the theorem actually implies the stronger form.

Shrinking $\overset{\circ}{\mathbb{D}}$ and $\mathcal{U}$ if necessary, let $\tilde{h} : \overset{\circ}{\mathbb{D}} \times \mathcal{U} \to \mathbb{C}$ denote a local extension of h to $\overset{\circ}{\mathbb{D}} \times \mathcal{U}$. We use w as our coordinate on $\overset{\circ}{\mathbb{D}}$. Note that replacing $h(w, \mathbf{z})$ by $h(w^2, \mathbf{z})$ does not change the statement of the theorem. Therefore, we can, and will, assume that $d_{(0,\mathbf{x})}\tilde{h}$ vanishes on $\mathbb{C} \times \{\mathbf{0}\}$.

Let $\tilde{p} : \overset{\circ}{\mathbb{D}} \times \mathcal{U} \to \mathcal{U}$ denote the projection, and let $p := \tilde{p}_{|_{\overset{\circ}{\mathbb{D}} \times X}}$. Let $\mathbf{Q}^\bullet := p^*\mathbf{P}^\bullet[1]$; as $\mathbf{P}^\bullet$ is perverse, so is $\mathbf{Q}^\bullet$. Let $Y := (\overset{\circ}{\mathbb{D}} \times X) \cap V(h - b)$, and let $\widehat{w} : Y \to \overset{\circ}{\mathbb{D}}$ denote the projection. Let $\mathbf{R}^\bullet := \mathbf{Q}^\bullet_{|_Y}[-1]$. Let $\hat{f} : Y \to \mathbb{C}$ be given by $\hat{f}(w, \mathbf{z}) := f(\mathbf{z})$. As we already know that the theorem is true for ordinary families of functions, we wish to apply it to the family of functions $\hat{f}_{\hat{w}}$ and the family of sheaves $\mathbf{R}^\bullet_{\hat{w}}$; this would clearly prove the desired result.

Thus, we need to prove two things: that $\mathbf{R}^\bullet$ is perverse near $(0, \mathbf{x})$, and that the family $\mathbf{R}^\bullet_{\hat{w}}$ is continuous at $(0, \mathbf{x})$.

Let $\{S_\alpha\}$ be a Whitney stratification, with connected strata, of X with respect to which $\mathbf{P}^\bullet$ is constructible. Refining the stratification if necessary, assume that $V(t - b)$ is a union of strata. Let $\mathrm{Ch}(\mathbf{P}^\bullet) = \sum m_\alpha \left[\overline{T^*_{S_\alpha}\mathcal{U}}\right]$. Clearly, $\mathbf{Q}^\bullet$ is constructible with respect to the Whitney stratification $\{\overset{\circ}{\mathbb{D}} \times S_\alpha\}$, and the characteristic cycle of $\mathbf{Q}^\bullet$ in $T^*(\overset{\circ}{\mathbb{D}} \times \mathcal{U})$ is given by $\mathrm{Ch}(\mathbf{Q}^\bullet) = -\sum m_\alpha \left[\overline{T^*_{\overset{\circ}{\mathbb{D}} \times S_\alpha}(\overset{\circ}{\mathbb{D}} \times \mathcal{U})}\right]$.

Note that, for all $(\mathbf{z}, \eta) \in T^*\mathcal{U}$, $(\mathbf{z}, \eta) \in \overline{T^*_{S_\alpha}\mathcal{U}}$ if and only if $(0, \mathbf{z}, \eta \circ d_{(0,\mathbf{z})}p) \in \overline{T^*_{\overset{\circ}{\mathbb{D}} \times S_\alpha}(\overset{\circ}{\mathbb{D}} \times \mathcal{U})}$. As we are assuming that $d_{(0,\mathbf{x})}\tilde{h}$ vanishes on $\mathbb{C} \times \{\mathbf{0}\}$ and that $h_0 = t$, we know that $d_{(0,\mathbf{x})}\tilde{h} = d_{\mathbf{x}}\tilde{t} \circ d_{(0,\mathbf{z})}\tilde{p}$. Thus, $(\mathbf{x}, d_{\mathbf{x}}\tilde{t}) \in \overline{T^*_{S_\alpha}\mathcal{U}}$ if and only if $(0, \mathbf{x}, d_{(0,\mathbf{x})}\tilde{h}) \in \overline{T^*_{\overset{\circ}{\mathbb{D}} \times S_\alpha}(\overset{\circ}{\mathbb{D}} \times \mathcal{U})}$. Therefore, $(\mathbf{x}, d_{\mathbf{x}}\tilde{t}) \in |\,\mathrm{Ch}(\mathbf{P}^\bullet)|$ if and only if $(0, \mathbf{x}, d_{(0,\mathbf{x})}\tilde{h}) \in |\,\mathrm{Ch}(\mathbf{Q}^\bullet)|$. As we are assuming that the family $\mathbf{P}^\bullet_t$ is continuous at $\mathbf{x}$, we may apply Lemma 5.3 to conclude that $(\mathbf{x}, d_{\mathbf{x}}\tilde{t}) \notin |\,\mathrm{Ch}(\mathbf{P}^\bullet)|$ and, hence, $(0, \mathbf{x}, d_{(0,\mathbf{x})}\tilde{h}) \notin |\,\mathrm{Ch}(\mathbf{Q}^\bullet)|$. It follows that, for all $(w, \mathbf{z})$ near $(0, \mathbf{x})$, $(w, \mathbf{z}, d_{(w,\mathbf{z})}\tilde{h}) \notin |\,\mathrm{Ch}(\mathbf{Q}^\bullet)|$ and that the family $\mathbf{Q}^\bullet_h$ is continuous at $(0, \mathbf{x})$; that is, there exists an open neighborhood, $\Omega \times \mathcal{W}$, of $(0, \mathbf{x})$ in $\overset{\circ}{\mathbb{D}} \times \mathcal{U}$, in which $\phi_{h-b}[-1]\mathbf{Q}^\bullet = 0$ and such that, if $(w, \mathbf{z}) \in \Omega \times \mathcal{W}$ and $m_\alpha \neq 0$, then $(w, \mathbf{z}, d_{(w,\mathbf{z})}\tilde{h}) \notin \overline{T^*_{\mathring{\mathbb{D}} \times S_\alpha}\big(\mathring{\mathbb{D}} \times \mathcal{U}\big)}$. For the remainder of the proof, we assume that $\mathring{\mathbb{D}}$ and $\mathcal{U}$ have been rechosen to be small enough to use for Ω and $\mathcal{W}$.

As $\phi_{h-b}[-1]\mathbf{Q}^\bullet = 0$, $\mathbf{R}^\bullet \cong \psi_{h-b}[-1]\mathbf{Q}^\bullet$ is a perverse sheaf on Y. It remains for us to show that the family $\mathbf{R}^\bullet_{\hat{w}}$ is continuous at $(0, \mathbf{x})$.

Of course, we appeal to Lemma 5.3 again – we need to show that $(0, \mathbf{x}, d_{(0,\mathbf{x})}w) \notin |\operatorname{Ch}(\mathbf{R}^\bullet)|$. Now, $|\operatorname{Ch}(\mathbf{R}^\bullet)| = |\operatorname{Ch}(\psi_{h-b}[-1]\mathbf{Q}^\bullet)|$, and we wish to use Theorem 2.3 to describe this characteristic variety. If $(w, \mathbf{z}) \in \Omega \times \mathcal{W}$ and $m_\alpha \neq 0$, then $(w, \mathbf{z}, d_{(w,\mathbf{z})}\tilde{h}) \notin \overline{T^*_{\mathring{\mathbb{D}} \times S_\alpha}\big(\mathring{\mathbb{D}} \times \mathcal{U}\big)}$; thus, if $m_\alpha \neq 0$, then h has no critical points when restricted to $\mathring{\mathbb{D}} \times S_\alpha$, and, using the notation of 2.2 and 2.3,

$$T^*_{h-b,\mathbf{Q}^\bullet}\big(\mathring{\mathbb{D}} \times \mathcal{U}\big) = \sum_\alpha m_\alpha \left[\overline{T^*_{h_{|_{\mathring{\mathbb{D}} \times S_\alpha}}}\big(\mathring{\mathbb{D}} \times \mathcal{U}\big)}\right].$$

Now, using Theorem 2.3, we find that

$$|\operatorname{Ch}(\mathbf{R}^\bullet)| = \big(V(h-b) \times \mathbb{C}^{n+2}\big) \cap \bigcup_{m_\alpha \neq 0} \overline{T^*_{h_{|_{\mathring{\mathbb{D}} \times S_\alpha}}}\big(\mathring{\mathbb{D}} \times \mathcal{U}\big)}.$$

We will be finished if we can show that, if $m_\alpha \neq 0$, then $(0, \mathbf{x}, d_{(0,\mathbf{x})}w) \notin \overline{T^*_{h_{|_{\mathring{\mathbb{D}} \times S_\alpha}}}\big(\mathring{\mathbb{D}} \times \mathcal{U}\big)}$.

Fix an S_α for which $m_\alpha \neq 0$. Suppose that $(0, \mathbf{x}, \eta) \in \overline{T^*_{h_{|_{\mathring{\mathbb{D}} \times S_\alpha}}}\big(\mathring{\mathbb{D}} \times \mathcal{U}\big)}$. Then, there exists a sequence $(w_i, \mathbf{z}_i, \eta_i) \in T^*_{h_{|_{\mathring{\mathbb{D}} \times S_\alpha}}}\big(\mathring{\mathbb{D}} \times \mathcal{U}\big)$ such that $(w_i, \mathbf{z}_i, \eta_i) \to (0, \mathbf{x}, \eta)$. Thus, $\eta_i\big((\mathbb{C} \times T_{\mathbf{z}_i}S_\alpha) \cap \ker d_{(w_i,\mathbf{z}_i)}\tilde{h}\big) = 0$. By taking a subsequence, if necessary, we may assume that $T_{\mathbf{z}_i}S_\alpha$ converges to some $\mathcal{T}$ in the appropriate Grassmanian. Now, we know that $\ker d_{(w_i,\mathbf{z}_i)}\tilde{h} \to \ker d_{(0,\mathbf{x})}\tilde{h} = \mathbb{C} \times \ker d_\mathbf{x}\tilde{t}$. As $(\mathbf{x}, d_\mathbf{x}\tilde{t}) \notin \overline{T^*_{S_\alpha}\mathcal{U}}$, $\mathbb{C} \times \ker d_\mathbf{x}\tilde{t}$ transversely intersects $\mathbb{C} \times \mathcal{T}$. Therefore, $(\mathbb{C} \times T_{\mathbf{z}_i}S_\alpha) \cap \ker d_{(w_i,\mathbf{z}_i)}\tilde{h} \to (\mathbb{C} \times \mathcal{T}) \cap (\mathbb{C} \times \ker d_\mathbf{x}\tilde{t})$, and so $\mathbb{C} \times \{\mathbf{0}\} \subseteq \ker \eta$. However, $\ker d_{(0,\mathbf{x})}w = \{0\} \times \mathbb{C}^{n+1}$, and we are finished. □

We would like to translate Theorem 5.7 into a statement about Milnor fibres and the constant sheaf. First, though, it will be convenient to prove a lemma.

Lemma 5.8. *Let $\mathbf{x} \in X$, and let $b := t(\mathbf{x})$. Suppose that $\dim_\mathbf{x}\big(V(t-b) \cap \overline{\Sigma_{\mathbb{C}}t}\big) \leqslant 0$. Fix an integer k. If $\widetilde{H}^k(F_{t,\mathbf{x}};\ \mathbb{C}) = 0$, then the family ${}^k\mathbf{P}^\bullet_t$ is continuous at $\mathbf{x}$. In addition, if $\widetilde{H}^k(F_{t,\mathbf{x}};\ \mathbb{C}) = 0$ and $\widetilde{H}^{k-1}(F_{t,\mathbf{x}};\ \mathbb{C}) = 0$, then ${}^k\mathbf{P}^\bullet_b \cong {}^pH^0(\mathbb{C}^\bullet_{X_b}[k])$ near $\mathbf{x}$.*

Proof. By Remark 1.7, the assumption that $\dim_\mathbf{x}\big(V(t-b) \cap \overline{\Sigma_{\mathbb{C}}t}\big) \leqslant 0$ is equivalent to $\dim_\mathbf{x}\overline{\Sigma_{\mathbb{C}}t} \leqslant 0$ and, by Theorem 3.6, this is equivalent to $\dim_\mathbf{x}\overline{\Sigma_{j\mathbf{P}^\bullet}t} \leqslant 0$ for all j. Thus, $\operatorname{supp}\phi_{t-b}[-1]^k\mathbf{P}^\bullet \subseteq \{\mathbf{x}\}$ near $\mathbf{x}$. We claim that the added assumption that $\widetilde{H}^k(F_{t,\mathbf{x}};\ \mathbb{C}) = 0$ implies that, in fact, $\phi_{t-b}[-1]^k\mathbf{P}^\bullet = 0$ near $\mathbf{x}$.

For, near $\mathbf{x}$, $\operatorname{supp}\phi_{t-b}[-1]\mathbb{C}^\bullet_X[k+1] \subseteq \{\mathbf{x}\}$, and so

$$\phi_{t-b}[-1]^k\mathbf{P}^\bullet = \phi_{t-b}[-1]^pH^0(\mathbb{C}^\bullet_X[k+1]) \cong {}^pH^0(\phi_{t-b}[-1]\mathbb{C}^\bullet_X[k+1]) \cong \mathbf{H}^0(\phi_{t-b}[-1]\mathbb{C}^\bullet_X[k+1]).$$

Near $\mathbf{x}$, $\phi_{t-b}[-1]\mathbb{C}^\bullet_X[k+1]$ is supported at, at most, the point $\mathbf{x}$ and, hence, $\phi_{t-b}[-1]^k\mathbf{P}^\bullet = 0$ provided that $\mathbf{H}^0(\phi_{t-b}[-1]\mathbb{C}^\bullet_X[k+1])_\mathbf{x} = 0$, i.e., provided that $\widetilde{H}^k(F_{t,\mathbf{x}};\ \mathbb{C}) = 0$. This proves the first claim in the lemma.

Now, if the family ${}^k\mathbf{P}^\bullet_t$ is continuous at $\mathbf{x}$, then, near $\mathbf{x}$,

$$ {}^k\mathbf{P}^\bullet_b = {}^k\mathbf{P}^\bullet_{|_{V(t-b)}}[-1] \cong \psi_{t-b}[-1]{}^pH^0(\mathbb{C}^\bullet_X[k+1]) \cong {}^pH^0(\psi_{t-b}[-1]\mathbb{C}^\bullet_X[k+1]), $$

and we claim that, if $\widetilde{H}^k(F_{t,\mathbf{x}};\ \mathbb{C}) = 0$ and $\widetilde{H}^{k-1}(F_{t,\mathbf{x}};\ \mathbb{C}) = 0$, then there is an isomorphism (in the derived category) ${}^pH^0(\psi_{t-b}[-1]\mathbb{C}^\bullet_X[k+1]) \cong {}^pH^0(\mathbb{C}^\bullet_{X_b}[k])$.

To see this, consider the canonical distinguished triangle

$$ \mathbb{C}^\bullet_{X_b}[k] \ \to\ \psi_{t-b}[-1]\mathbb{C}^\bullet_X[k+1] \ \to\ \phi_{t-b}[-1]\mathbb{C}^\bullet_X[k+1] \ \xrightarrow{[1]}\ \mathbb{C}^\bullet_{X_b}[k]. $$

A portion of the long exact sequence (in the category of perverse sheaves) resulting from applying perverse cohomology is given by

$$ {}^pH^{-1}(\phi_{t-b}[-1]\mathbb{C}^\bullet_X[k+1]) \to {}^pH^0(\mathbb{C}^\bullet_{X_b}[k]) \to {}^pH^0(\psi_{t-b}[-1]\mathbb{C}^\bullet_X[k+1]) \to {}^pH^0(\phi_{t-b}[-1]\mathbb{C}^\bullet_X[k+1]). $$

We would be finished if we knew that the terms on both ends of the above were zero. However, since $\phi_{t-b}[-1]\mathbb{C}^\bullet_X[k+1]$ has no support other than $\mathbf{x}$ (near $\mathbf{x}$), we proceed as we did above to show that ${}^pH^{-1}(\phi_{t-b}[-1]\mathbb{C}^\bullet_X[k+1])$ and ${}^pH^0(\phi_{t-b}[-1]\mathbb{C}^\bullet_X[k+1])$ are zero precisely when $\widetilde{H}^{k-1}(F_{t,\mathbf{x}};\ \mathbb{C})$ and $\widetilde{H}^k(F_{t,\mathbf{x}};\ \mathbb{C})$ are zero. □

Theorem 5.9. *Let $\mathbf{x} \in X$ and let $b := t(\mathbf{x})$. Suppose that $\mathbf{x} \notin \overline{\Sigma_{\mathbb{C}} t}$, and that $\dim_{\mathbf{x}} \Sigma_{\mathbb{C}}(f_b) \leqslant 0$.*

Then, there exists a neighborhood, $\mathcal{W}$, of $\mathbf{x}$ in X such that, for all a near b, there are a finite number of points $\mathbf{y} \in \mathcal{W} \cap V(t-a)$ for which $\widetilde{H}^*(F_{f_a,\mathbf{y}};\ \mathbb{C}) \neq 0$; moreover, for all integers, k, $\tilde{b}_{k-1}(F_{f_a,\mathbf{y}}) = \mu_{\mathbf{y}}(f_a; {}^k\mathbf{P}^\bullet_a)$, and

$$ \tilde{b}_{k-1}(F_{f_b,\mathbf{x}}) = \sum_{\mathbf{y}\in\mathcal{W}\cap V(t-a)} \tilde{b}_{k-1}(F_{f_a,\mathbf{y}}), $$

where $\widetilde{H}^*()$ and $\tilde{b}_*()$ are as in Remark 3.4.

Proof. Let $v := f_b(\mathbf{x})$. Fix an integer k.

By the lemma, the family ${}^k\mathbf{P}^\bullet_t$ is continuous at $\mathbf{x}$ and ${}^k\mathbf{P}^\bullet_b \cong {}^pH^0(\mathbb{C}^\bullet_{X_b}[k])$ near $\mathbf{x}$. Thus,

$$ \phi_{f_b-v}[-1]{}^k\mathbf{P}^\bullet_b \cong \phi_{f_b-v}[-1]{}^pH^0(\mathbb{C}^\bullet_{X_b}[k]) \cong {}^pH^0(\phi_{f_b-v}[-1]\mathbb{C}^\bullet_{X_b}[k]). $$

We are assuming that $\dim_{\mathbf{x}} \Sigma_{\mathbb{C}}(f_b) \leqslant 0$; this is equivalent to: $\operatorname{supp}\phi_{f_b-v}[-1]\mathbb{C}^\bullet_{X_b}[k] \subseteq \{\mathbf{x}\}$ near $\mathbf{x}$, it follows from the above line and Theorem 3.6 that $\dim_{\mathbf{x}} \Sigma_{{}^k\mathbf{P}^\bullet_b} f_b \leqslant 0$ and that

$$ (\ddagger) \qquad \mu_{\mathbf{x}}(f_b; {}^k\mathbf{P}^\bullet_b) = \dim H^0(\phi_{f_b-v}[-1]\mathbb{C}^\bullet_{X_b}[k])_{\mathbf{x}} = \tilde{b}_{k-1}(F_{f_b,\mathbf{x}}). $$

Applying Theorem 5.7, we find that there exists an open neighborhood $\mathcal{W}'$ of $\mathbf{x}$ in $\mathcal{U}$ such that, for all $\mathbf{y} \in \mathcal{W}'$, if $a := t(\mathbf{y})$, then $(*)$ $\dim_{\mathbf{y}} \Sigma_{{}^k\mathbf{P}^\bullet_a} f_a \leqslant 0$, and, for fixed a close to b, there are a finite number of points $\mathbf{y} \in \mathcal{W}' \cap V(t-a)$ such that $\mu_{\mathbf{y}}(f_a; {}^k\mathbf{P}^\bullet_a) \neq 0$ and

$$ (\dagger) \qquad \mu_{\mathbf{x}}(f_b; {}^k\mathbf{P}^\bullet_b) = \sum_{\mathbf{y}\in\mathcal{W}\cap V(t-a)} \mu_{\mathbf{y}}(f_a; {}^k\mathbf{P}^\bullet_a). $$

Now, using the above argument for all k with $0 \leqslant k \leqslant \dim X - 1$ and intersecting the resulting $\mathcal{W}'$-neighborhoods, we obtain an open neighborhood $\mathcal{W}$ of $\mathbf{x}$ such that $(*)$ and $(\dagger)$ hold for all such k. We claim that, if a is close to b, then $\mathcal{W}\cap\overline{\Sigma_{\mathbb{C}}f_a}$ consists of isolated points, i.e., the points $\mathbf{y}\in\mathcal{W}\cap V(t-a)$ for which $\widetilde{H}^*(F_{f_a,\mathbf{y}};\ \mathbb{C})\neq 0$ are isolated.

If $a=b$, then there is nothing to show. So, assume that $a\neq b$, and assume that we are working in $\mathcal{W}$ throughout. By Remark 1.7, t satisfies the hypotheses of Lemma 5.8 at $t=a$; hence, for all k, not only is ${}^k\mathbf{P}^\bullet_t$ continuous at $t=a$, but we also know that ${}^k\mathbf{P}^\bullet_a\cong {}^pH^0(\mathbb{C}^\bullet_{X_a}[k])$. By Theorem 3.6, $\overline{\Sigma_{\mathbb{C}}f_a}=\bigcup\overline{\Sigma_{{}^k\mathbf{P}^\bullet_a}f_a}$, where the union is over k where $0\leqslant k\leqslant\dim X_a$. As $\dim X_a\leqslant\dim X-1$, the claim follows from $(*)$ and the definition of $\mathcal{W}$.

Now that we know that ${}^k\mathbf{P}^\bullet_t$ is continuous at $t=a$ and that $\mathcal{W}\cap\overline{\Sigma_{\mathbb{C}}f_a}$ consists of isolated points, we may use the argument that produced (‡) to conclude that $\mu_{\mathbf{y}}(f_a;{}^k\mathbf{P}^\bullet_a)=\tilde{b}_{k-1}(F_{f_a,\mathbf{y}})$. The theorem follows from this, (‡), $(*)$, and (†). □

We want to prove a result which generalizes that of Lê and Saito [**L-S**]. We need to make the assumption that the Milnor number is constant along a curve that is embedded in X. Hence, it will be convenient to use a *local section* of $t: X\to\mathbb{C}$ at a point $\mathbf{x}\in X$; that is, an analytic function $\mathbf{r}$ from an open neighborhood, $\mathcal{V}$, of $t(\mathbf{x})$ in $\mathbb{C}$ into X such that $\mathbf{r}(t(\mathbf{x}))=\mathbf{x}$ and $t\circ\mathbf{r}$ equals the inclusion morphism of $\mathcal{V}$ into $\mathbb{C}$. Note that existence of such a local section implies that $\mathbf{x}\notin\Sigma_{\mathrm{alg}}t$; in particular, $V(\tilde{t}-\tilde{t}(\mathbf{x}))$ is smooth at $\mathbf{x}$.

Theorem 5.10. *Suppose that the family $\mathbf{P}^\bullet_t$ is continuous at $\mathbf{x}\in X$. Let $b:=t(\mathbf{x})$, and let $v:=f_b(\mathbf{x})$. Let $\mathbf{r}:\mathcal{V}\to X$ be a local section of t at $\mathbf{x}$, and let $C:=\operatorname{im}\mathbf{r}$. Assume that $C\subseteq V(f-v)$, that $\dim_{\mathbf{x}}\Sigma_{\mathbf{P}^\bullet_b}f_b\leqslant 0$, and that, for all a close to b, the Milnor number $\mu_{\mathbf{r}(a)}(f_a;\mathbf{P}^\bullet_a)$ is non-zero and is independent of a; denote this common value by μ.*

Then, C is smooth at $\mathbf{x}$, $V(\tilde{t}-b)$ transversely intersects C in $\mathcal{U}$ at $\mathbf{x}$, and there exists a neighborhood, $\mathcal{W}$, of $\mathbf{x}$ in X such that $\mathcal{W}\cap\overline{\Sigma_{\mathbf{P}^\bullet}f}\subseteq C$ and $\left(\phi_{f-v}[-1]\mathbf{P}^\bullet\right)_{|\mathcal{W}\cap C}\cong\left(\mathbb{C}^\mu_{\mathcal{W}\cap C}[1]\right)^\bullet$. In particular, if we let $\hat{t}$ denote the restriction of t to $V(f-v)$, then the family $\left(\phi_{f-v}[-1]\mathbf{P}^\bullet\right)_{\hat{t}}$ is continuous at $\mathbf{x}$.

If, in addition to the other hypotheses, we assume that $\mathbf{x}\in\Sigma_{\mathrm{alg}}f$, then the two families $\left(\psi_{f-v}[-1]\mathbf{P}^\bullet\right)_{\hat{t}}$ and $\left(\mathbf{P}^\bullet_{|_{V(f-v)}}[-1]\right)_{\hat{t}}$ are continuous at $\mathbf{x}$. (Though $\mathbf{P}^\bullet_{|_{V(f-v)}}[-1]$ need not be perverse.)

Proof. Let us first prove that the last statement of the theorem follows easily from the first portion of the theorem. So, assume that $\phi_{\hat{t}-b}[-1]\phi_{f-v}[-1]\mathbf{P}^\bullet=0$ near $\mathbf{x}$. Therefore, working near $\mathbf{x}$, we have that $\phi_{\hat{t}-b}[-1]\left(\mathbf{P}^\bullet_{|_{V(f-v)}}[-1]\right)\cong\phi_{\hat{t}-b}[-1]\psi_{f-v}[-1]\mathbf{P}^\bullet$, and we need to show that this is the zero-sheaf. By Lemma 5.3, what we need to show is that $(\mathbf{x},d_{\mathbf{x}}\tilde{t})\notin|\operatorname{Ch}(\psi_{f-v}[-1]\mathbf{P}^\bullet)|=|\operatorname{Ch}(\psi_{f-v}\mathbf{P}^\bullet)|$. As we are assuming that $\mathbf{x}\in\Sigma_{\mathrm{alg}}f$, we may apply Corollary 4.5 to find that it suffices to show that $(\mathbf{x},d_{\mathbf{x}}\tilde{t})\notin|\operatorname{Ch}(\phi_{f-v}\mathbf{P}^\bullet)|=|\operatorname{Ch}(\phi_{f-v}[-1]\mathbf{P}^\bullet)|$. By 5.3, this is equivalent to $\phi_{\hat{t}-b}[-1]\phi_{f-v}[-1]\mathbf{P}^\bullet=0$ near $\mathbf{x}$, which we already know to be true. This proves the last statement of the theorem.

Before proceeding with the remainder of the proof, we wish to make some simplifying assumptions. As $\mathbf{x}\notin\Sigma_{\mathrm{alg}}t$, we may certainly perform an analytic change of coordinates in $\mathcal{U}$ to reduce ourselves to the case where t is simply the restriction to X of a linear form $\tilde{t}$. Moreover, it is notational convenient to assume, without loss of generality, that $\mathbf{x}=\mathbf{0}$ and that b and v are both zero.

Let $\{S_\alpha\}$ be a Whitney stratification of X with connected strata with respect to which $\mathbf{P}^\bullet$ is constructible and such that $V(t)$ and $V(f)$ are each unions of strata. Suppose that $\operatorname{Ch}(\mathbf{P}^\bullet)$ is given by $\sum_\alpha m_\alpha\left[\overline{T^*_{S_\alpha}\mathcal{U}}\right]$.

Let $\widetilde{C}:=\{(\mathbf{r}(a),d_{\mathbf{r}(a)}\tilde{f})\mid a\in\mathcal{V}\}$; the projection, ρ, onto the first component induces an isomorphism from $\widetilde{C}$ to C. By Lemma 5.6, the assumption that the Milnor number, $\mu_{\mathbf{r}(a)}(f_a; {}^k\mathbf{P}^\bullet_a)$, is independent of a is equivalent to:

(†) there exists an open neighborhood $\widetilde{\mathcal{W}}$ of $(\mathbf{0}, d_{\mathbf{0}}\tilde{f})$ in $T^*\mathcal{U}$ in which $\widetilde{C}$ equals

$$\operatorname{im} d\tilde{f} \ \cap \bigcup_{m_\alpha \neq 0} \overline{T^*_{t_{|_{S_\alpha}}}\mathcal{U}}$$

and $\widetilde{C}$ is a smooth curve at $(\mathbf{0}, d_{\mathbf{0}}\tilde{f})$ such that $(\mathbf{0}, d_{\mathbf{0}}\tilde{f}) \notin \Sigma(t \circ \rho_{|_{\widetilde{C}}})$.

It follows immediately that C is smooth at $\mathbf{0}$ and $\mathbf{0} \notin \Sigma(t_{|_C})$. We need to show that (†) implies that $\mathcal{W} \cap \overline{\Sigma_{\mathbf{P}^\bullet} f} \subseteq C$ and $\left(\phi_f[-1]\mathbf{P}^\bullet\right)_{|_{\mathcal{W}\cap C}} \cong \left(\mathbb{C}^\mu_{\mathcal{W}\cap C}[1]\right)^\bullet$, where $\mathcal{W} := \rho(\widetilde{\mathcal{W}})$.

As $\overline{T^*_{S_\alpha}\mathcal{U}} \subseteq \overline{T^*_{t_{|_{S_\alpha}}}\mathcal{U}}$, we have that $|\operatorname{Ch}(\mathbf{P}^\bullet)| \subseteq \bigcup_{m_\alpha \neq 0} \overline{T^*_{t_{|_{S_\alpha}}}\mathcal{U}}$ and, thus, $\operatorname{im} d\tilde{f} \cap |\operatorname{Ch}(\mathbf{P}^\bullet)| \subseteq \widetilde{C}$ inside $\widetilde{\mathcal{W}}$. It follows from Theorem 3.2 that $\mathcal{W} \cap \overline{\Sigma_{\mathbf{P}^\bullet} f} \subseteq C$.

It remains for us to show that $\left(\phi_f[-1]\mathbf{P}^\bullet\right)_{|_{\mathcal{W}\cap C}} \cong \left(\mathbb{C}^\mu_{\mathcal{W}\cap C}[1]\right)^\bullet$. As $\phi_f[-1]\mathbf{P}^\bullet$ is perverse and we have just shown that the support of $\phi_f[-1]\mathbf{P}^\bullet$, near $\mathbf{0}$, is a smooth curve, it follows from the work of MacPherson and Vilonen in [**M-V**] that what we need to show is that, for a generic linear form L, $\mathbf{Q}^\bullet := \phi_L[-1]\phi_f[-1]\mathbf{P}^\bullet = 0$ near $\mathbf{0}$. By definition of the characteristic cycle (and since $\mathbf{0}$ is an isolated point in the support of $\mathbf{Q}^\bullet$), this is the same as showing that the coefficient of $T^*_{\{\mathbf{0}\}}\mathcal{U}$ in $\operatorname{Ch}(\phi_f[-1]\mathbf{P}^\bullet)$ equals zero. To show this, we will appeal to Theorem 2.4 and use the notation from there.

We need to show that $m_{\mathbf{0}}(\phi_f[-1]\mathbf{P}^\bullet) = 0$. By 2.4, if suffices to show that $m_{\mathbf{0}}(\mathbf{P}^\bullet) = 0$ and $\Gamma^1_{f,L}(S_\alpha) = \emptyset$ near $\mathbf{0}$, for all S_α which are not contained in $V(f)$ and for which $m_\alpha \neq 0$ (where L still denotes a generic linear form). As $\mathbf{P}^\bullet_t$ is continuous at $\mathbf{0}$, Lemma 5.3 tells us that $m_{\mathbf{0}}(\mathbf{P}^\bullet) = 0$. Now, near $\mathbf{0}$, if $\mathbf{y} \in \Gamma^1_{f,L}(S_\alpha) - \{\mathbf{0}\}$, then $(\mathbf{y}, d_{\mathbf{y}}\tilde{f}) \in T^*_{L_{|_{S_\alpha}}}\mathcal{U}$. If we knew that, near $(\mathbf{0}, d_{\mathbf{0}}\tilde{f})$, $\widetilde{C}$ equals $\operatorname{im} d\tilde{f} \cap \bigcup_{m_\alpha \neq 0} \overline{T^*_{L_{|_{S_\alpha}}}\mathcal{U}}$, then we would be finished – for C is contained in $V(f)$ while S_α is not; hence, $\Gamma^1_{f,L}(S_\alpha)$ would have to be empty near $\mathbf{0}$.

Looking back at (†), we see that what we still need to show is that if $\widetilde{C}$ equals $\operatorname{im} d\tilde{f} \ \cap \bigcup_{m_\alpha \neq 0} \overline{T^*_{t_{|_{S_\alpha}}}\mathcal{U}}$ near $(\mathbf{0}, d_{\mathbf{0}}\tilde{f})$, then the same statement holds with t replaced by a generic linear form L. We accomplish this by perturbing t until it is generic, and by then showing that this perturbed t satisfies the hypotheses of the theorem.

As C is smooth and transversely intersected by $V(\tilde{t})$ at $\mathbf{0}$, by performing an analytic change of coordinates, we may assume that $\tilde{t} = z_0$, that C is the z_0-axis, and that $r(a) = (a, \mathbf{0})$. Since the set of linear forms for which 2.4 holds is generic, there exists an open disk, $\overset{\circ}{\mathbb{D}}$, around the origin in $\mathbb{C}$ and an analytic family $\tilde{h} : (\overset{\circ}{\mathbb{D}} \times \mathcal{U}, \overset{\circ}{\mathbb{D}} \times \{\mathbf{0}\}) \to (\mathbb{C}, 0)$ such that $\tilde{h}_0(\mathbf{z}) := \tilde{h}(0, \mathbf{z}) = \tilde{t}(\mathbf{z})$ and such that, for all small non-zero c, $\tilde{h}_c(\mathbf{z}) := \tilde{h}(c, \mathbf{z})$ is a linear form for which Theorem 2.4 holds. Let $h := \tilde{h}_{|_{\overset{\circ}{\mathbb{D}} \times X}}$.

As the family $\mathbf{P}^\bullet_t$ is continuous at $\mathbf{0}$, Lemma 5.5 tells us that $\mathbf{P}^\bullet_{h_c}$ is continuous at $\mathbf{0}$ for all small c. As we are now considering these two different families with the same underlying sheaf, the expression $\mathbf{P}^\bullet_a$ for a fixed value of a is ambiguous, and we need to adopt some new notation. We continue to let $\mathbf{P}^\bullet_a := \mathbf{P}^\bullet_{|_{V(t-a)}}[-1]$ and $f_a := f_{|_{V(t-a)}}$, and let ${}_c\mathbf{P}^\bullet_a := \mathbf{P}^\bullet_{|_{V(h_c-a)}}[-1]$ and ${}_cf_a := f_{|_{V(h_c-a)}}$.

Since $V(\tilde{h}_0) = V(z_0)$ transversely intersects C at $\mathbf{0}$ in $\mathcal{U}$, for all small c, $V(h_c)$ transversely intersects C at $\mathbf{0}$ in $\mathcal{U}$. Hence, for all small c, there exists a local section $\mathbf{r}_c(a)$ for h_c at $\mathbf{0}$ such that $\operatorname{im} \mathbf{r}_c \subseteq C$.

We claim that, for all small c:

i) $\dim_{\mathbf{0}} \Sigma_{{}_c\mathbf{P}^\bullet_0}({}_cf_0) \leqslant 0$ and $\mu_{\mathbf{0}}({}_cf_0;\ {}_c\mathbf{P}^\bullet_0) \leqslant \mu_{\mathbf{0}}({}_0f_0;\ {}_0\mathbf{P}^\bullet_0) = \mu_{\mathbf{0}}(f_0;\ \mathbf{P}^\bullet_0)$;

ii) for all small a, $\dim_{\mathbf{r}_c(a)} \Sigma_{{}_c\mathbf{P}^\bullet_a}({}_cf_a) \leqslant 0$ and $\mu_{\mathbf{r}_c(a)}({}_cf_a;\ {}_c\mathbf{P}^\bullet_a) \leqslant \mu_{\mathbf{0}}({}_0f_0;\ {}_0\mathbf{P}^\bullet_0)$; and

iii) for all small $a \neq 0$, $\mu_{\mathbf{r}_c(a)}({}_cf_a;\ {}_c\mathbf{P}^\bullet_a) = \mu_{\mathbf{r}_c(a)}(f_{z_0(\mathbf{r}_c(a))};\ \mathbf{P}^\bullet_{z_0(\mathbf{r}_c(a))})$.

Note that proving i), ii), and iii) would complete the proof of the theorem, for they imply that the hypotheses of the theorem hold with t replaced by h_c for all small c. To be precise, we would know that $\mathbf{P}^\bullet_{h_c}$ is continuous at $\mathbf{0}$, $\dim_{\mathbf{0}} \Sigma_{{}_c\mathbf{P}^\bullet_0}({}_cf_0) \leqslant 0$, and, for all small a, $\mu_{\mathbf{r}_c(a)}({}_cf_a;\ {}_c\mathbf{P}^\bullet_a) = \mu_{\mathbf{0}}({}_cf_0;\ {}_c\mathbf{P}^\bullet_0)$; this last equality follows from i), ii), and iii), since, for all small $a \neq 0$, we would have

$$\mu = \mu_{\mathbf{r}_c(a)}(f_{z_0(\mathbf{r}_c(a))};\ \mathbf{P}^\bullet_{z_0(\mathbf{r}_c(a))}) = \mu_{\mathbf{r}_c(a)}({}_cf_a;\ {}_c\mathbf{P}^\bullet_a) \leqslant \mu_{\mathbf{0}}({}_cf_0;\ {}_c\mathbf{P}^\bullet_0) \leqslant \mu_{\mathbf{0}}(f_0;\ \mathbf{P}^\bullet_0) = \mu.$$

However, i), ii) and iii) are easy to prove. i) and ii) follow immediately from Theorem 5.7, and iii) follows simply from the fact that, for all small $a \neq 0$, $V(z_0 - z_0(\mathbf{r}_c(a)))$ and $V(\tilde{h}_c - \tilde{h}_c(\mathbf{r}_c(a)))$ are smooth and transversely intersect all strata of any analytic stratification of X in a neighborhood of $(0, \mathbf{0})$. This concludes the proof. □

Corollary 5.11. *Suppose that the family $\mathbf{P}^\bullet_t$ is continuous at $\mathbf{x} \in X$. Let $b := t(\mathbf{x})$, and let $v := f_b(\mathbf{x})$. Let $\mathbf{r} : \mathcal{V} \to X$ be a local section of t at $\mathbf{x}$, and let $C := \operatorname{im} \mathbf{r}$. Assume that $C \subseteq V(f - v)$, that $\dim_{\mathbf{x}} \Sigma_{\mathbf{P}^\bullet_b} f_b \leqslant 0$, and that, for all a close to b, the Milnor number $\mu_{\mathbf{r}(a)}(f_a; \mathbf{P}^\bullet_a)$ is non-zero and is independent of a. Let $\operatorname{Ch}(\mathbf{P}^\bullet) = \sum_\alpha m_\alpha \left[\overline{T^*_{S_\alpha}\mathcal{U}}\right]$, where $\{S_\alpha\}$ is a collection of connected analytic submanifolds of $\mathcal{U}$.*

Then, C is smooth at $\mathbf{x}$, and there exists a neighborhood, $\mathcal{W}$, of $\mathbf{x}$ in X such that, for all S_α for which $S_\alpha \not\subseteq V(f - v)$ and $m_\alpha \neq 0$:

$\mathcal{W} \cap \Sigma\big(f_{|_{(\overline{S_\alpha})_{\mathrm{reg}}}}\big) \subseteq C$ and, if $\mathbf{x} \in \Sigma_{\mathrm{Nash}}(f_{|_{\overline{S}_\alpha}})$, then the pair $\Big(\big(\overline{S_\alpha}\big)_{\mathrm{reg}},\ C\Big)$ satisfies Thom's a_f condition at $\mathbf{x}$.

Proof. One applies Theorem 5.10. The fact that $\mathcal{W} \cap \Sigma\big(f_{|_{(\overline{S_\alpha})_{\mathrm{reg}}}}\big) \subseteq C$, for all S_α for which $m_\alpha \neq 0$ follows from Theorem 3.2, since $\mathcal{W} \cap \overline{\Sigma_{\mathbf{P}^\bullet} f} \subseteq C$. The remainder of the corollary follows by applying Corollary 4.4, where one uses C for the submanifold N. □

Just as we used perverse cohomology to translate Theorem 5.7 into a statement about the constant sheaf in Theorem 5.9, we can use perverse cohomology to translate Corollary 5.11. We will use the notation and results from Proposition 3.3 and Remark 3.4.

Corollary 5.12. *Let $b := t(\mathbf{x})$, and let $v := f_b(\mathbf{x})$. Suppose that $\mathbf{x} \notin \overline{\Sigma_{\mathbb{C}} t}$. Suppose, further, that, $\dim_{\mathbf{x}} \Sigma_{\mathbb{C}}(f_b) \leqslant 0$.*

Let $\mathbf{r} : \mathcal{V} \to X$ be a local section of t at $\mathbf{x}$, and let $C := \operatorname{im} \mathbf{r}$. Assume that $C \subseteq V(f - v)$.

Let S_α be a visible stratum of X of dimension d_α, not contained in $V(f - v)$, and let j be an integer such that $\tilde{b}_{j-1}(\mathbb{L}_\alpha) \neq 0$. Let $Y := \overline{S_\alpha}$ and let $k := d_\alpha + j - 1$. In particular, Y could be any irreducible component of X, j could be zero, and k would be $(\dim Y) - 1$.

Suppose that the reduced Betti number $\tilde{b}_{k-1}(F_{f_a, \mathbf{r}(a)})$ is independent of a for all small a, and that either

a) $\mathbf{x} \in \Sigma_{\mathrm{Nash}}(f_{|_Y})$; or that

b) $\mathbf{x} \notin \Sigma_{\mathrm{cnr}}(f_{|_Y})$, C is smooth at $\mathbf{x}$, and (Y_{reg}, C) satisfies Whitney's condition a) at $\mathbf{x}$.

Then, C is smooth at $\mathbf{x}$, and the pair $(Y_{\mathrm{reg}},\ C)$ satisfies the a_f condition at $\mathbf{x}$.

Moreover, in case a), $\tilde{b}_{k-1}(F_{f_a, \mathbf{r}(a)}) \neq 0$, C is transversely intersected by $V(\tilde{t} - b)$ at $\mathbf{x}$, and $\Sigma(f_{|_{Y_{\mathrm{reg}}}}) \subseteq C$ near $\mathbf{x}$.

In addition, if $\mathbf{x} \in \Sigma_{\rm alg} f$ *and, for all small* a *and for all* i, $\tilde{b}_i(F_{f_a, \mathbf{r}(a)})$ *is independent of* a, *then* $\mathbf{x} \notin \overline{\Sigma_{\mathbb{C}}(t_{|_{V(f-v)}})}$.

Proof. We will dispose of case b) first. Suppose that $\mathbf{x} \notin \Sigma_{\rm cnr}(f_{|_Y})$, C is smooth at $\mathbf{x}$, and $(Y_{\rm reg}, C)$ satisfies Whitney's condition a) at $\mathbf{x}$. Let $\overset{\circ}{Y} := Y_{\rm reg} - \Sigma(f_{|_{Y_{\rm reg}}})$.

Suppose that we have an analytic path $(\mathbf{x}(t), \eta_t) \in \overline{T^*_{f_{|_{\overset{\circ}{Y}}}}\mathcal{U}}$, where $(\mathbf{x}(0), \eta_0) = (\mathbf{x}, \eta)$ and, for $t \neq 0$, $(\mathbf{x}(t), \eta_t) \in T^*_{f_{|_{\overset{\circ}{Y}}}}\mathcal{U}$. We wish to show that $(\mathbf{x}, \eta) \in T^*_C\mathcal{U}$.

For $t \neq 0$, $\mathbf{x}(t) \in \overset{\circ}{Y}$, and thus η_t can be written uniquely as $\eta_t = \omega_t + \lambda_t d_{\mathbf{x}_t}\tilde{f}$, where $\omega_t \in T^*_{\overset{\circ}{Y}}\mathcal{U}$ and $\lambda_t \in \mathbb{C}$. As we saw in Theorem 4.2, this implies that either $|\lambda_t| \to \infty$ or that $\lambda_t \to \lambda_0$, for some $\lambda_0 \in \mathbb{C}$. If $|\lambda_t| \to \infty$, then $\frac{\eta_t}{\lambda_t} \to 0$ and, therefore, $-\frac{\omega_t}{\lambda_t} \to d_{\mathbf{x}}\tilde{f}$; however, this implies that $\mathbf{x} \in \Sigma_{\rm cnr}(f_{|_Y})$, contrary to our assumption. Thus, we must have that $\lambda_t \to \lambda_0$.

It follows at once that ω_t converges to some ω_0. By Whitney's condition a), $(\mathbf{x}, \omega_0) \in T^*_C\mathcal{U}$. As $C \subseteq V(f-v)$, $(\mathbf{x}, d_{\mathbf{x}}\tilde{f}) \in T^*_C\mathcal{U}$. Hence, $(\mathbf{x}, \eta) \in T^*_C\mathcal{U}$ and we have finished with case b).

We must now prove the results in case a). The main step is to prove that $\tilde{b}_{k-1}(F_{f_b, \mathbf{x}}) \neq 0$.

We may refine our stratification, if necessary, so that $V(t-b)$ is a union of strata. By the first part of Theorem 5.9, $\tilde{b}_{k-1}(F_{f_b, \mathbf{x}}) = \mu_{\mathbf{x}}(f_b; {}^k\mathbf{P}^\bullet_b)$. Hence, by Lemma 5.6.iii, $\tilde{b}_{k-1}(F_{f_b, \mathbf{x}})$ would be unequal to zero if we knew, for some S_β for which $m_\beta({}^k\mathbf{P}^\bullet) \neq 0$, that $(\mathbf{x}, d_{\mathbf{x}}\tilde{f}) \in \overline{T^*_{t_{|_{S_\beta}}}\mathcal{U}}$. However, our fixed S_α is such a stratum, for $b_{k+1-d_\alpha}(N_\alpha, \mathbb{L}_\alpha) \neq 0$ and, since $\mathbf{x} \in \Sigma_{\rm Nash}(f_{|_Y})$, $\mathbf{x} \in \Sigma_{\rm cnr}(f_{|_Y})$ and so $(\mathbf{x}, d_{\mathbf{x}}\tilde{f}) \in \overline{T^*_{S_\alpha}\mathcal{U}} \subseteq \overline{T^*_{t_{|_{S_\alpha}}}\mathcal{U}}$.

Now, applying the first part of 5.9 again, we have that $\mu_{\mathbf{r}(a)}(f_a;\ {}^k\mathbf{P}^\bullet_a) = \tilde{b}_{k-1}(F_{f_a, \mathbf{r}(a)})$ for all small a. The conclusions in case a) follow from Corollary 5.11.

We must still demonstrate the last statement of corollary.

Suppose that if $\tilde{b}_i(F_{f_a, \mathbf{r}(a)})$ is independent of a for all small a and for all i. Let $\hat{t}$ denote the restriction of t to $V(f-v)$. We will work in a small neighborhood of $\mathbf{x}$. Applying the last two sentences of Theorem 5.10, we find that $\phi_{\hat{t}-b}[-1]\phi_{f-v}[-1]{}^i\mathbf{P}^\bullet = 0$ and $\phi_{\hat{t}-b}[-1]\psi_{f-v}[-1]{}^i\mathbf{P}^\bullet = 0$ for all i. Commuting nearby and vanishing cycles with perverse cohomology, we find that

$$ {}^pH^0\big(\phi_{\hat{t}-b}[-1]\phi_{f-v}[-1]\mathbb{C}^\bullet_X[i+1]\big) = 0 \qquad \text{and} \qquad {}^pH^0\big(\phi_{\hat{t}-b}[-1]\psi_{f-v}[-1]\mathbb{C}^\bullet_X[i+1]\big) = 0, $$

for all i. Therefore, $\phi_{\hat{t}-b}[-1]\phi_{f-v}[-1]\mathbb{C}^\bullet_X = 0$ and $\phi_{\hat{t}-b}[-1]\psi_{f-v}[-1]\mathbb{C}^\bullet_X = 0$. It follows from the existence of the distinguished triangle (relating nearby cycles, vanishing cycles, and restriction to the hypersurface) that $\phi_{\hat{t}-b}[-1]\mathbb{C}^\bullet_{V(f-v)}[-1] = 0$. This proves the last statement of the corollary. □

Remark 5.13. If X is a connected l.c.i., then each $\mathbb{L}_\alpha$ has (possibly) non-zero cohomology concentrated in middle degree. Hence, for each visible S_α, $\tilde{b}_{j-1}(\mathbb{L}_\alpha) \neq 0$ only when $j = \operatorname{codim}_X S_\alpha$; this corresponds to $k = (\dim X) - 1$. Therefore, the degree $(\dim X) - 2$ reduced Betti number of $F_{f_a, \mathbf{r}(a)}$ controls the a_f condition between **all** visible strata and C.

Corollary 5.14. *Let* W *be an analytic subset of an open subset of* $\mathbb{C}^n$. *Let* Z *be a* d*-dimensional irreducible component of* W. *Let* $X := \overset{\circ}{\mathbb{D}} \times W$ *be the product of an open disk about the origin with* W, *and let* $Y := \overset{\circ}{\mathbb{D}} \times Z$. *Let* $f : (X, \overset{\circ}{\mathbb{D}} \times \{\mathbf{0}\}) \to (\mathbb{C}, 0)$ *be an analytic function, such that* $f_{|_Y} \not\equiv 0$, *and let* $f_t(\mathbf{z}) := f(t, \mathbf{z})$.

Suppose that $\mathbf{0}$ *is an isolated point of* $\Sigma_{\mathbb{C}}(f_0)$, *and that the reduced Betti number* $\tilde{b}_{d-1}(F_{f_a,(a,\mathbf{0})})$ *is independent of* a *for all small* a.

If either a) $\mathbf{0} \in \Sigma_{\text{Nash}}(f_{|_Y})$ *or* b) $\mathbf{0} \notin \Sigma_{\text{cnr}}(f_{|_Y})$, *then the pair* $(Y_{\text{reg}}, \overset{\circ}{\mathbb{D}} \times \{\mathbf{0}\})$ *satisfies Thom's* a_f *condition at* $\mathbf{0}$.

Moreover, in case a), $\tilde{b}_{d-1}(F_{f_a,(a,\mathbf{0})}) \neq 0$ *and, near* $\mathbf{0}$, $\Sigma(f_{|_{Y_{\text{reg}}}}) \subseteq \overset{\circ}{\mathbb{D}} \times \{\mathbf{0}\}$.

Remark 5.15. A question naturally arises: how effective is the criterion appearing in Corollary 5.14 that $\tilde{b}_{d-1}(F_{f_a,(a,\mathbf{0})})$ is independent of a?

By Proposition 3.10, if $\{R_\beta\}$ is a Whitney stratification of W, then (using the notation from 3.10)

$$\tilde{b}_{d-1}(F_{f_a,(a,\mathbf{0})}) =$$

$$\tilde{b}_{d-1}(\mathbb{L}_{\{\mathbf{0}\}}) \quad + \sum_{\substack{R_\beta \text{ visible} \\ \dim R_\beta \geqslant 1}} \tilde{b}_{d-1-d_\beta}(\mathbb{L}_\beta) \Big(\big(\Gamma^1_{f_a,L}(R_\beta) \cdot V(f_a)\big)_{\mathbf{0}} - \big(\Gamma^1_{f_a,L}(R_\beta) \cdot V(L)\big)_{\mathbf{0}} \Big),$$

where $\mathbb{L}_{\{\mathbf{0}\}}$ denotes the complex link of the origin. As the Betti numbers do not vary with a, $\tilde{b}_{d-1}(F_{f_a,(a,\mathbf{0})})$ will be independent of a provided that $\big(\Gamma^1_{f_a,L}(R_\beta) \cdot V(f_a)\big)_{\mathbf{0}} - \big(\Gamma^1_{f_a,L}(R_\beta) \cdot V(L)\big)_{\mathbf{0}}$ is independent of a for all visible strata, R_β, of dimension at least one.

This condition is certainly very manageable to check if the dimension of the singular set of X at the origin is zero or one.

The final statement of Corollary 5.12 has as its conclusion that the constant sheaf on $X \cap V(f - v)$, parametrized by the restriction of t, is continuous at $\mathbf{x}$; this is useful for inductive arguments, since the hypothesis on the ambient space in Corollary 5.12 is that the constant sheaf, parametrized by t, should be continuous at $\mathbf{x}$. For instance, we can prove the following corollary.

Corollary 5.16. *Suppose that* $f^1, \dots, f^k$ *are analytic functions from* $\mathcal{U}$ *into* $\mathbb{C}$ *which define a sequence of local complete intersections at the origin, i.e., are such that, for all* i *with* $1 \leqslant i \leqslant k$, *the space* $X^{n+1-i} := V(f^1, \dots, f^i)$ *is a local complete intersection of dimension* $n+1-i$ *at the origin. If, for all* i, X_t^{n+1-i} *has an isolated singularity at the origin and the restrictions* $f_t^{i+1} : X_t^{n+1-i} \to \mathbb{C}$ *are such that* $\dim_{\mathbf{0}} \Sigma_{\text{can}} f_t^{i+1} \leqslant 0$ *and have Milnor numbers (in the sense of* [**Lo**]*) which are independent of* t, *then* $\Sigma\big(f_{|_{X_{\text{reg}}^{n+1-(k-1)}}}\big) \subseteq \mathbb{C} \times \{\mathbf{0}\}$ *and the pair* $\big(X_{\text{reg}}^{n+1-(k-1)}, \mathbb{C} \times \{\mathbf{0}\}\big)$ *satisfies the* a_{f^k} *condition at the origin.*

Proof. Recall that $\mathbb{C}^\bullet_X[\dim X]$ is a perverse sheaf if X is a local complete intersection. The "ordinary" Milnor number of f_t^{i+1} at the origin is equal to $\mu_{\mathbf{0}}(f_t^{i+1};\ \mathbb{C}^\bullet_{X_t^{n+1-i}}[n-i])$. Hence, using Proposition 3.10.ii, this Milnor number is equal to the degree $n-i-1$ (the "middle" degree) reduced Betti number of the Milnor fibre of f_t^{i+1} at the origin – the only possible non-zero reduced Betti number. Now, use Corollary 5.12 and induct; the inductive requirement on the Milnor fibre of z_0 follows from the last statement of the corollary. □

Remark 5.17. In [**G-K**], Gaffney and Kleiman deal with families of local complete intersections as above. In this setting, they obtain the result of Corollary 5.16 using multiplicities of modules.

§6. **Concluding remarks**.

We hope to have convinced the reader that the correct notion of "the critical locus" of a function, f, on a singular space is given by $\Sigma_{\mathbb{C}} f$.

We also hope to have convinced the reader of (at least) three other things: that the vanishing cycles control Thom's a_f condition (as demonstrated in Corollary 4.4, Corollary 5.11, and Corollary 5.12), that the correct setting to be in to generalize many classical results is where one uses arbitrary perverse sheaves as coefficients, and that perverse cohomology is an amazing tool for turning statements about perverse sheaves into statements about the constant sheaf.

While a great deal of material concerning local complete intersections appears in the singularities literature, it is not so easy to find results that apply to arbitrary analytic spaces. As we remarked earlier, from our point of view, what is special about l.c.i.'s is that the shifted constant sheaf is perverse; this implies that the reduced cohomology of the links of Whitney strata are concentrated in middle degree. As we discussed in Example 3.12, this allows us to algebraically calculate the Betti numbers of the links. In the case of a general space, the obstruction to algebraically calculating Milnor numbers is that there is no general algebraic manner for calculating the Betti numbers of the links of strata.

Finally, we wish to say a few words about future directions for our work. In [**Ma2**], [**Ma3**], and [**Ma4**], we developed the Lê cycles and Lê numbers of an affine hypersurface singularity. These Lê numbers appear to be the "correct" generalization of the Milnor number to the case of arbitrary, non-isolated, affine hypersurface singularities. Now, in this paper, we have generalized the Milnor number to the case of isolated hypersurface singularities on an arbitrary analytic space. By combining these two approaches, we can obtain a super generalization of the Milnor number – one that works for arbitrary analytic functions on arbitrary analytic spaces. Moreover, using this generalization, we can prove a super generalization of the result of Lê and Saito in [**L-S**]. Of course, as we discussed above, the problem of actually calculating these generalized Milnor-Lê numbers is precisely the problem of calculating the Betti numbers of the complex links of Whitney strata.

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