

Enriched Relative Polar Curves and Discriminants

David B. Massey

In honor of Lê Dũng Tráng on his 60th birthday

Abstract

Let (f, g) be a pair of complex analytic functions on a singular analytic space X . We give “the correct” definition of the relative polar curve of (f, g) , and we prove a very general form of Lê’s attaching result, which relates the relative polar curve to the relative cohomology of the Milnor fiber modulo a hyperplane slice. Our main theorem is the construction, in non-generic situations, of a derived category version of the discriminant and Cerf diagram of a pair of functions. From this, we derive a number of generalizations of results which are classically proved using the discriminant. In particular, we give applications to families of isolated “critical points”.

1 Introduction

Let \mathcal{U} be an open neighborhood of the origin in \mathbb{C}^{n+1} , and let $\tilde{f} : \mathcal{U} \rightarrow \mathbb{C}$ be a complex analytic function. We assume that $\mathbf{0} \in V(\tilde{f}) := \tilde{f}^{-1}(0)$. We let $\Sigma\tilde{f}$ denote the critical locus of \tilde{f} .

Fix a point $p \in \mathcal{U}$. Let z_0 denote a generic linear form on \mathbb{C}^{n+1} , which, in fact, we take as the first coordinate function, after possibly performing a generic linear change of coordinates.

In [4], [23], [7], [8], Hamm, Teissier, and Lê define and use the relative polar curve (of \tilde{f} with respect to z_0), $\Gamma_{\tilde{f}, z_0}^1$, to prove a number of topological results related to the Milnor fiber $F_{\tilde{f}, \mathbf{0}}$ of a hypersurface singularity. We shall recall some definitions and results here. We should mention that there are a number of different characterizations of the relative polar, all of which agree when z_0 is sufficiently generic; below, we have selected what we consider the easiest way of describing the relative polar curve as a set, a scheme, and a cycle.

As a set, $\Gamma_{\tilde{f}, z_0}^1$ is the closure of the critical locus of (\tilde{f}, z_0) minus the critical locus of \tilde{f} , i.e., $\Gamma_{\tilde{f}, z_0}^1$ equals $\overline{\Sigma(\tilde{f}, z_0) - \Sigma\tilde{f}}$, as a set. If z_0 is sufficiently generic for \tilde{f} at p , then, in a neighborhood of p , $\Gamma_{\tilde{f}, z_0}^1$ will be purely one-dimensional (which includes the possibility of being empty); see Theorem 1.1 below.

It is not difficult to give $\Gamma_{\tilde{f}, z_0}^1$ a scheme structure. We use (z_0, \dots, z_n) as coordinates on \mathcal{U} . If $\Gamma_{\tilde{f}, z_0}^1$ is purely one-dimensional at p , then, at points x near, but unequal to, p , $\Gamma_{\tilde{f}, z_0}^1$ is given the structure of the scheme $V\left(\frac{\partial\tilde{f}}{\partial z_1}, \dots, \frac{\partial\tilde{f}}{\partial z_n}\right)$. One can also remove “algebraically” any embedded components of $\Gamma_{\tilde{f}, z_0}^1$ at p by using *gap sheaves*; see Chapter 1 of [12].

¹The author would like to thank Tsukuba University, Tokyo Science University, Hokkaido University, and K. Takeuchi, M. Oka, and T. Ohmoto; Section 6 of this paper was written during a visit to these universities with the support of these mathematicians.

AMS subject classifications 32B15, 32C35, 32C18, 32B10.

keywords: polar curve, discriminant, Milnor fiber, nearby cycles, a_f condition.

In practice, all topological applications of the relative polar curve use only its structure as an analytic cycle (germ), that is, as a locally finite sum of irreducible analytic sets (or germs of sets) counted with integral multiplicities (which will all be non-negative). We remark here that these are cycles, **not** cycle classes; we do not mean up to rational equivalence. The intersection theory that one needs here is the simple case of proper intersections inside smooth manifolds; see 8.2 of [2] or our summary in Appendix A of [17]. If C is a one-dimensional irreducible germ of $\Gamma_{\tilde{f}, z_0}^1$ at p , and $x \in C$ is close to, but unequal to, p , then the component C appears in the cycle $\Gamma_{\tilde{f}, z_0}^1$ with multiplicity given by the Milnor number of $\tilde{f}|_H$ at x , where H is a generic affine hyperplane passing through x .

The following theorem tells one the relative polar curve has nice properties for a generic choice of the linear form z_0 .

Theorem 1.1. (Hamm-Lê) *For a generic choice of z_0 ,*

1. $\Gamma_{\tilde{f}, z_0}^1$ *is purely one-dimensional at p ;*
2. $\Gamma_{\tilde{f}, z_0}^1$ *properly intersects $V(\tilde{f} - \tilde{f}(p))$ at p , i.e., p is an isolated point in $\Gamma_{\tilde{f}, z_0}^1 \cap V(\tilde{f} - \tilde{f}(p))$;*
3. *the cycle $\Gamma_{\tilde{f}, z_0}^1$ is reduced (near p), i.e., each component through p appears with multiplicity 1.*

Proof. Items 1 and 2 are proved in 2.1 of [4]. Item 3 is Lemma 2.2.1 of [4]. \square

Now, we can state the main result of [7].

Theorem 1.2. (Lê, [7]) *For generic z_0 , up to homotopy, $F_{\tilde{f}, \mathbf{0}}$ is obtained from $F_{\tilde{f}|_{V(z_0)}, \mathbf{0}}$ by attaching $\tau := (\Gamma_{\tilde{f}, z_0}^1 \cdot V(f))_{\mathbf{0}}$ n -cells.*

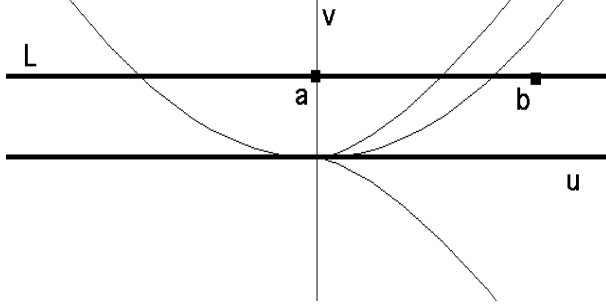
In particular, $H^k(F_{\tilde{f}, \mathbf{0}}, F_{\tilde{f}|_{V(z_0)}, \mathbf{0}}) = 0$ if $k \neq n$, and $H^n(F_{\tilde{f}, \mathbf{0}}, F_{\tilde{f}|_{V(z_0)}, \mathbf{0}}) \cong \mathbb{Z}^\tau$.

Remark 1.3. It will be important for us to understand some of Lê's set-up in [7].

Let B_ϵ denote a closed $2n$ -ball of radius ϵ centered at the origin in \mathbb{C}^n , and let \mathbb{D}_δ denote a closed disk of radius δ centered at the origin in \mathbb{C} .

Then, Lê shows that, for $0 < |\xi| \ll \delta \ll \epsilon \ll 1$, $(\mathbb{D}_\delta \times B_\epsilon) \cap \tilde{f}^{-1}(\xi)$ has the homotopy-type of the Milnor fiber of f at $\mathbf{0}$ and, of course, $(\{0\} \times B_\epsilon) \cap \tilde{f}^{-1}(\xi)$ is homeomorphic to the Milnor fiber, $F_{\tilde{f}_0, \mathbf{0}}$, of $\tilde{f}_0 := \tilde{f}|_{V(z_0)}$ at $\mathbf{0}$. Theorem 1.2 is obtained by applying Morse Theory to the map $|z_0|^2$ on $(\mathbb{D}_\delta \times B_\epsilon) \cap \tilde{f}^{-1}(\xi)$.

In [7], Lê gives an extensive discussion of the discriminant and Cerf diagram of the map $G := (z_0, \tilde{f})$. The discriminant is $G(\Sigma G)$ and the Cerf diagram is $G(\Gamma_{\tilde{f}, z_0}^1)$. One "sees" Lê's attaching result graphically in the discriminant/Cerf diagram below. Down in the image of the map (z_0, \tilde{f}) , the pair $(F_{\tilde{f}, \mathbf{0}}, F_{\tilde{f}_0, \mathbf{0}})$ is represented by the pair $(L, \{a\})$, and one sees that the relative cohomology of $(F_{\tilde{f}, \mathbf{0}}, F_{\tilde{f}_0, \mathbf{0}})$ decomposes as a direct sum of local Morse data above each of the points where L intersects the Cerf diagram, C , (the image of the relative polar curve). Now, the number of points in the intersection of C and L , counted with multiplicities, is precisely $\tau = (\Gamma_{\tilde{f}, z_0}^1 \cdot V(f))_{\mathbf{0}}$, which yields the theorem.



From Lê's discussion, it is clear that $F_{\tilde{f}_0, \mathbf{0}}$ is also homeomorphic to $\widehat{F}_{\tilde{f}_0, \mathbf{0}} := (\{\nu\} \times B_\epsilon) \cap \tilde{f}^{-1}(\xi)$, provided that $0 < |\xi| \ll \nu < \delta \ll \epsilon \ll 1$; in fact, there is a homeomorphism from the pair $(F_{\tilde{f}, \mathbf{0}}, F_{\tilde{f}_0, \mathbf{0}})$ to $(F_{\tilde{f}, \mathbf{0}}, \widehat{F}_{\tilde{f}_0, \mathbf{0}})$ which induces an isomorphism between $H^*(F_{\tilde{f}, \mathbf{0}}, F_{\tilde{f}_0, \mathbf{0}})$ and $H^*(F_{\tilde{f}, \mathbf{0}}, \widehat{F}_{\tilde{f}_0, \mathbf{0}})$ and which induces the identity map on $H^*(F_{\tilde{f}, \mathbf{0}})$. In the Cerf diagram, the pair $(F_{\tilde{f}, \mathbf{0}}, \widehat{F}_{\tilde{f}_0, \mathbf{0}})$ is represented by $(L, \{b\})$ and, using an argument which is essentially the same as in the paragraph above, one concludes that $H^*(F_{\tilde{f}, \mathbf{0}}, \widehat{F}_{\tilde{f}_0, \mathbf{0}}) \cong \mathbb{Z}^\tau$.

This is an important observation, because in the formalism of the derived category and vanishing cycles, $H^{k+1}(F_{\tilde{f}, \mathbf{0}}, \widehat{F}_{\tilde{f}_0, \mathbf{0}})$ is isomorphic to the stalk cohomology at the origin of the vanishing cycles along z_0 of the nearby cycles along f of the constant sheaf on \mathcal{U} , i.e., $H^{k+1}(F_{\tilde{f}, \mathbf{0}}, \widehat{F}_{\tilde{f}_0, \mathbf{0}}) \cong H^k(\phi_{z_0} \psi_f \mathbb{Z}_{\mathcal{U}}^\bullet)_{\mathbf{0}}$ (here, we do not distinguish between z_0 and $z_0|_{V(f)}$). If we include the correct shifts, then we know that $\mathbb{Z}_{\mathcal{U}}^\bullet[n+1]$ is perverse, and that $\phi_{z_0}[-1]$ and $\psi_f[-1]$ take perverse sheaves to perverse sheaves; hence, we prefer to write $H^k(\phi_{z_0}[-1] \psi_f[-1] \mathbb{Z}_{\mathcal{U}}^\bullet[n+1])_{\mathbf{0}} \cong H^{k+n}(F_{\tilde{f}, \mathbf{0}}, \widehat{F}_{\tilde{f}_0, \mathbf{0}})$.

Thus, the results of Lê in [7] tell one that $H^k(\phi_{z_0}[-1] \psi_f[-1] \mathbb{Z}_{\mathcal{U}}^\bullet[n+1])_{\mathbf{0}}$ is zero, unless $k = 0$, and $H^0(\phi_{z_0}[-1] \psi_f[-1] \mathbb{Z}_{\mathcal{U}}^\bullet[n+1])_{\mathbf{0}} \cong \mathbb{Z}^\tau$.

The above relation between Theorem 1.2 and iterated vanishing and nearby cycles appeared explicitly in the work of Sabbah in [22] and in our own work in [13].

We now wish to describe one of our primary results from [13]; this result is a substantial generalization of Theorem 1.2.

Let X be a closed analytic subspace of \mathcal{U} , and let $f := \tilde{f}|_X$. Let $\tilde{g} : (\mathcal{U}, \mathbf{0}) \rightarrow (\mathbb{C}, 0)$ be another analytic function, and let $g := \tilde{g}|_X$. Let \mathfrak{S}° be a Whitney stratification, with connected strata, of $X - V(f)$. For $S \in \mathfrak{S}^\circ$, let \mathbb{N}_S and \mathbb{L}_S denote, respectively, the normal slice and link of the stratum S ; see [3]. Let \mathbf{F}^\bullet be a bounded complex of sheaves of \mathbb{Z} -modules on $X - V(f)$, constructible with respect to \mathfrak{S}° . Let d_S denote the dimension of $S \in \mathfrak{S}^\circ$.

It would take far too long to define some of the objects and terms in the statement below. Let us simply say that $\Gamma_{f,g}(\mathfrak{S}^\circ)$ is generalization of the underlying set of the relative polar curve and, when this set is, in fact, 1-dimensional, we give it a cycle structure and denote the portion of the cycle coming from a stratum $S \in \mathfrak{S}^\circ$ by $\tilde{\Gamma}_{f,g}^1(S)$. The terms *tractable* and *decent* are non-degeneracy requirements on g ; their definitions depend on the choice of a *good stratification* (i.e., an a_f stratification). In the classical situation where \tilde{g} is a generic linear form, it is easy to see that $\Gamma_{f,g}(\mathfrak{S}^\circ)$ is 1-dimensional, and that g is tractable and decent.

In [13], we proved:

Theorem 1.4. ([13], Theorem 4.2) *Suppose that g is tractable relative to f at $\mathbf{0}$.*

Then, for all i , $\mathbb{H}^i(F_{f,\mathbf{0}}, F_{f|_{V(g)}}, \mathbf{0}; \mathbf{F}^\bullet)$ is a direct summand of $H^{i-1}(\phi_g \psi_f \mathbf{F}^\bullet)_{\mathbf{0}}$, and there exist integers j_S such that

$$\mathbb{H}^i(F_{f,\mathbf{0}}, F_{f|_{V(g)}}, \mathbf{0}; \mathbf{F}^\bullet) \cong \bigoplus_{S \in \mathfrak{S}^\circ} (\mathbb{H}^{i-d_S+1}(\mathbb{N}_S, \mathbb{L}_S; \mathbf{F}^\bullet))^{j_S},$$

where $j_S \geq (\tilde{\Gamma}_{f,g}^1(S) \cdot V(f))_{\mathbf{0}}$, with equality if g is decent relative to f .

Furthermore, if $\Gamma_{f,g}(\mathfrak{S}^\circ)$ has no components contained in $V(g)$, then

$$H^{i-1}(\phi_g \psi_f \mathbf{F}^\bullet)_{\mathbf{0}} \cong \mathbb{H}^i(F_{f,\mathbf{0}}, F_{f|_{V(g)}}, \mathbf{0}; \mathbf{F}^\bullet).$$

Summary of the Results of this Paper

What are the problems with Theorem 1.4? There are several. One is that the hypotheses are difficult to check. Another, related, problem is that it is unclear to what extent the hypotheses are necessary for the conclusion. A third issue is that the definition of $\tilde{\Gamma}_{f,g}^1(S)$ seems rather ad hoc.

In this paper, we “fix” these problems. Let \mathfrak{S}° and \mathbf{F}^\bullet be as above. Let $\mathfrak{S}^\circ(\mathbf{F}^\bullet)$ be the set of strata of \mathfrak{S} such that $\mathbb{H}^*(\mathbb{N}_S, \mathbb{L}_S; \mathbf{F}^\bullet) \neq 0$.

For each $S \in \mathfrak{S}^\circ(\mathbf{F}^\bullet)$, we will define an (ordinary) cycle $\Gamma_{f,\tilde{g}}(S)$. Using these cycles, we will define (Definition 3.4) the *graded, enriched relative polar cycle*, $(\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet))^\bullet$. In each degree $k \in \mathbb{Z}$, $(\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet))^k$ is a formal, locally finite, sum of irreducible analytic subsets of X multiplied by modules over a fixed base ring; see Section 2 of [18] and Section 3. When the underlying set, $|(\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet))^\bullet| = \bigcup_{S \in \mathfrak{S}^\circ(\mathbf{F}^\bullet)} |\Gamma_{f,\tilde{g}}(S)|$ is purely one-dimensional at a point $p \in X$, we say that *the relative polar curve of f , with respect to g , with coefficients in \mathbf{F}^\bullet , is defined at p* . A principal theme of this paper is that this definition of the relative polar curve is **the** correct definition for results on the level of cohomology.

There are notions of the critical loci of f and (f, g) with respect to the complex of sheaves \mathbf{F}^\bullet ; see Definition 4.2, Definition 4.6, and Remark 4.7. These critical loci are denoted by $\Sigma_{\mathbf{F}^\bullet} f$ and $\Sigma_{\mathbf{F}^\bullet}(f, g)$, respectively. As in the classical case, one has

$$\overline{\Sigma_{\mathbf{F}^\bullet}(f, g)} = \overline{\Sigma_{\mathbf{F}^\bullet} f} \cup |\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)|,$$

and that, in a neighborhood of the origin, $\overline{\Sigma_{\mathbf{F}^\bullet} f} \subseteq V(f)$. **Unlike** the classical case where \tilde{g} is a generic linear form, it is possible for $|\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)|$ to have components contained in $\overline{\Sigma_{\mathbf{F}^\bullet} f}$ and, hence, in $V(f)$. It is of fundamental importance throughout this paper that the precise amount of genericity that we need in our hypotheses is that $|\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)|$ is 1-dimensional and has no components contained in $V(f)$.

We should remark that the intersection product that we use throughout our work is a mild extension of the intersection theory, mentioned above, of properly intersecting cycles in a complex manifold; see Section 2 of [18] for the fundamental properties. We use \odot to denote this enriched intersection product.

Using our results in [18], and continuing with the notation from above, we will quickly prove our first main theorem. Below, and throughout this paper, we adopt the convention that the empty set has dimension $-\infty$, so that for analytic space Z and a point p , the condition that $\dim_p Z \leq 0$ means that either $\dim_p Z = 0$ or $p \notin Z$.

Main Theorem 1. (Theorem 3.12) *In a neighborhood of the origin,*

$$\text{supp } \phi_g[-1]\psi_f[-1]\mathbf{F}^\bullet = V(f) \cap |\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)| \subseteq V(g),$$

and, when $\dim_{\mathbf{0}} V(f) \cap |\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)| \leq 0$,

$$H^k(\phi_g[-1]\psi_f[-1]\mathbf{F}^\bullet)_{\mathbf{0}} \cong ((\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet))^k \odot V(f))_{\mathbf{0}},$$

i.e.,

$$H^{k-1}(B_\epsilon \cap f^{-1}(\xi), B_\epsilon \cap f^{-1}(\xi) \cap g^{-1}(\nu); \mathbf{F}^\bullet) \cong \bigoplus_{S \in \mathfrak{S}^\circ(\mathbf{F}^\bullet)} (\mathbb{H}^{k-d_S}(\mathbb{N}_S, \mathbb{L}_S; \mathbf{F}^\bullet))^{j_S},$$

where $0 < |\xi| \ll |\nu| \ll \epsilon \ll 1$, and $j_S = (\Gamma_{f,g}^1(S) \cdot V(f))_{\mathbf{0}}$.

Our proof of the above theorem is elegant, and very short, given existing results. However, it is not as intuitive as the discriminant/Cerf diagram argument, nor does it allow us to prove a number of related results, as we did in Section 4 of [13].

Hence, in Section 4 of this paper, we will prove the necessary technical results to push-down the complex \mathbf{F}^\bullet , restricted to a suitable neighborhood, via the map (g, f) . This will tell us that, if we define a derived category version of the discriminant, $\Delta_{\mathbf{F}^\bullet}(g, f)$, to be the image under (g, f) of the set $\overline{\Sigma_{\mathbf{F}^\bullet}(f, g)}$, then the standard classical discriminant/Cerf diagram proofs of Lê, Hamm, and Teissier will work without modification, and will yield hypercohomology results. It is important that the neighborhood $N_{\delta, \rho}^\epsilon$ below is defined using a **closed** ball, for it makes the map $T_{\delta, \rho}^\epsilon$ a proper map. The difficulty is in showing that, despite the fact that $N_{\delta, \rho}^\epsilon$ is not a complex analytic set, the proper push-forward of the restriction of \mathbf{F}^\bullet to $N_{\delta, \rho}^\epsilon$ is nonetheless complex analytically constructible.

For $\epsilon, \delta, \rho > 0$, let $N_{\delta, \rho}^\epsilon := B_\epsilon \cap g^{-1}(\mathring{\mathbb{D}}_\delta) \cap f^{-1}(\mathring{\mathbb{D}}_\rho)$. Let $(\mathbf{F}^\bullet)_{\delta, \rho}^\epsilon$ be the restriction of \mathbf{F}^\bullet to $N_{\delta, \rho}^\epsilon$, and let $T_{\delta, \rho}^\epsilon$ be the restriction of the map (g, f) to a map from $N_{\delta, \rho}^\epsilon$ to $\mathring{\mathbb{D}}_\delta \times \mathring{\mathbb{D}}_\rho$. We prove:

Main Theorem 2. (Theorem 4.13) Suppose that $\dim_{\mathbf{0}} V(f) \cap |\Gamma_{f, \bar{g}}(\mathbf{F}^\bullet)| \leq 0$.

Then, for all sufficiently small $\epsilon > 0$, there exist $\delta, \rho > 0$ such that the derived push-forward $R(T_{\delta, \rho}^\epsilon)_*(\mathbf{F}^\bullet)_{\delta, \rho}^\epsilon$ is complex analytically constructible with respect to the stratification given by

$$\{\mathring{\mathbb{D}}_\delta \times \mathring{\mathbb{D}}_\rho - \Delta_{\mathbf{F}^\bullet}(g, f), (\mathring{\mathbb{D}}_\delta \times \mathring{\mathbb{D}}_\rho) \cap \Delta_{\mathbf{F}^\bullet}(g, f) - \{\mathbf{0}\}, \{\mathbf{0}\}\}.$$

That **some** theorem like Theorem 4.13 is true is stunningly unsurprising. However, the technical details of dealing with boundary issues under the weak hypothesis that $\dim_{\mathbf{0}} V(f) \cap |\Gamma_{f, \bar{g}}(\mathbf{F}^\bullet)| \leq 0$ are immensely complicated. The usefulness of this theorem is that, after one has the technical proof, in the future, one avoids having to prove results about compatible neighborhoods of various forms under various technical hypotheses.

As an example of the usefulness of Theorem 4.13, we quickly prove:

Corollary to Main Theorem 2. (Corollary 4.18) Suppose that $\dim_{\mathbf{0}} V(f) \cap |\Gamma_{f, \bar{g}}(\mathbf{F}^\bullet)| \leq 0$. Let $(\widehat{\Gamma}_{f, \bar{g}}(\mathbf{F}^\bullet))^\bullet$ denote the components of $(\Gamma_{f, \bar{g}}(\mathbf{F}^\bullet))^\bullet$ which are not contained in $V(g)$. Then,

1.

$$\mathbb{H}^{k-1}(F_{f, \mathbf{0}}, F_{f|_{V(g)}, \mathbf{0}}; \mathbf{F}^\bullet) \cong ((\widehat{\Gamma}_{f, \bar{g}}(\mathbf{F}^\bullet))^k \odot V(f))_{\mathbf{0}};$$

2.

$$\mathbb{H}^{k-1}(F_{g, \mathbf{0}}, F_{g|_{V(f)}, \mathbf{0}}; \mathbf{F}^\bullet) \cong ((\widehat{\Gamma}_{f, \bar{g}}(\mathbf{F}^\bullet))^k \odot V(g))_{\mathbf{0}}; \text{ and}$$

3.

$$H^k(\phi_f[-1]\psi_g[-1]\mathbf{F}^\bullet)_{\mathbf{0}} \cong ((\widehat{\Gamma}_{f, \bar{g}}(\mathbf{F}^\bullet))^k \odot V(g))_{\mathbf{0}} \oplus H^k(\psi_g[-1]\phi_f[-1]\mathbf{F}^\bullet)_{\mathbf{0}}.$$

In Section 5, we give another application of our results. We combine the results of Section 4 with Corollary 3.9 of [19] in order to obtain a relation between Thom's a_f condition and the graded, enriched polar curve. In Section 6, we show how the main theorems allow us to prove a number of familiar-looking results on families with isolated critical points.

We belatedly thank Marc Levine for a number of helpful discussions involving our enriched intersection theory.

2 Basics of Enriched Cycles

In this section, we will recall the basic definitions that one needs for using enriched cycles ; these definitions are taken from Section 2 of [18]. There are a number of results from [18] which will be used in the proof of the main theorem in Section 3. While we will not restate the needed results from [18] in this paper, the background material in this section will enable the reader to make sense of the definition of the graded, enriched relative polar curve and the proof of the main theorem in Section 3.

Definition 2.1. *An enriched cycle, E , in X is a formal, locally finite sum $\sum_V E_V[V]$, where the V 's are irreducible analytic subsets of X and the E_V 's are finitely-generated R -modules. We refer to the V 's as the components of E , and to E_V as the V -component module of E . Two enriched cycles are considered the same provided that all of the component modules are isomorphic. The underlying set of E is $|E| := \cup_{E_V \neq 0} V$.*

If $C = \sum n_V[V]$ is an ordinary positive cycle in X , i.e., all of the n_v are non-negative integers, then there is a corresponding enriched cycle $[C]^{\text{enr}}$ in which the V -component module is the free R -module of rank n_V . If R is an integral domain, so that rank of an R -module is well-defined, then an enriched cycle E yields an ordinary cycle $[E]^{\text{ord}} := \sum_V (\text{rk}(E_V))[V]$.

If q is a finitely-generated module and E is an enriched cycle, then we let $qE := \sum_V (q \otimes E_V)[V]$; thus, if R is an integral domain and E is an enriched cycle, $[qE]^{\text{ord}} = (\text{rk}(q))[E]^{\text{ord}}$ and if C is an ordinary positive cycle and n is a positive integer, then $[nC]^{\text{enr}} = R^n[C]^{\text{enr}}$.

The (direct) sum of two enriched cycles D and E is given by $(D + E)_V := D_V \oplus E_V$.

There is a partial ordering on enriched cycles given by: $D \leq E$ if and only if there exists an enriched cycle P such that $D + P = E$. This relation is clearly reflexive and transitive; moreover, anti-symmetry follows from the fact that if M and N are Noetherian modules such that $M \oplus N \cong M$, then $N = 0$.

If two irreducible analytic subsets V and W intersect properly in \mathcal{U} , then the (ordinary) intersection cycle $[V] \cdot [W]$ is a well-defined positive cycle; we define the enriched intersection product of $[V]^{\text{enr}}$ and $[W]^{\text{enr}}$ by $[V]^{\text{enr}} \odot [W]^{\text{enr}} = ([V] \cdot [W])^{\text{enr}}$. If D and E are enriched cycles, and every component of D properly intersects every component of E in \mathcal{U} , then we say that D and E intersect properly in \mathcal{U} and we extend the intersection product linearly, i.e., if $D = \sum_V D_V[V]$ and $E = \sum_W E_W[W]$, then

$$D \odot E := \sum_{V,W} (D_V \otimes E_W)([V] \cdot [W])^{\text{enr}}.$$

A graded, enriched cycle E^\bullet is simply an enriched cycle E^i for i in some bounded set of integers. A single enriched cycle is considered as a graded enriched cycle by being placed totally in degree zero. The analytic set V is a component of E^\bullet if and only if V is a component of E^i for some i , and the underlying set of E^\bullet is $|E^\bullet| = \cup_i |E^i|$. If R is a domain, then E^\bullet yields an ordinary cycle $[E^\bullet]^{\text{ord}} := \sum_i (-1)^i (\text{rk}(E_V^i))[V]$. If k is an integer, we define the k -shifted graded, enriched cycle $E^\bullet[k]$ by $(E^\bullet[k])^i := E^{i+k}$.

If q is a finitely-generated module and E^\bullet is a graded enriched cycle, then we define the graded enriched cycle qE^\bullet by $(qE^\bullet)^i := \sum_V (q \otimes E_V^i)[V]$. The (direct) sum of two graded enriched cycles D^\bullet and E^\bullet is given

by $(D^\bullet + E^\bullet)_V^i := D_V^i \oplus E_V^i$. If D^i properly intersects E^j for all i and j , then we say that D^\bullet and E^\bullet intersect properly and we define the intersection product by

$$(D^\bullet \odot E^\bullet)^k := \sum_{i+j=k} (D^i \odot E^j).$$

Whenever we use the enriched intersection product symbol, we mean that we are considering the objects on both sides of \odot as graded, enriched cycles, even if we do not superscript by enr or \bullet .

Let $\tau : W \rightarrow Y$ be a proper morphism between analytic spaces. If $C = \sum n_V [V]$ is an ordinary positive cycle in W , then the proper push-forward $\tau_*(C) = \sum n_V \tau_*([V])$ is a well-defined ordinary cycle.

Definition 2.2. If $E^\bullet = \sum_V E_V^\bullet[V]$ is an enriched cycle in W , then we define the **proper push-forward of E^\bullet by τ** to be the graded enriched cycle $\tau_*^\bullet(E^\bullet)$ defined by

$$\tau_*^\bullet(E^\bullet) := \sum_V E_V^j[\tau_*([V])]^{\text{enr}}.$$

The ordinary projection formula for divisors ([F], 2.3.c) immediately implies the following enriched version.

Proposition 2.3. Let E^\bullet be a graded enriched cycle in X . Let $W := |E^\bullet|$. Let $\tau : W \rightarrow Y$ be a proper morphism, and let $g : Y \rightarrow \mathbb{C}$ be an analytic function such that $g \circ \tau$ is not identically zero on any component of E^\bullet . Then, g is not identically zero on any component of $\tau_*^\bullet(E^\bullet)$ and

$$\tau_*^\bullet(E^\bullet \odot V(g \circ \tau)) = \tau_*^\bullet(E^\bullet) \odot V(g).$$

Definition 2.4. Suppose that \mathbf{F}^\bullet is a bounded complex of sheaves, which is constructible with respect to an analytic Whitney stratification \mathfrak{S} , in which the strata are connected. For $S \in \mathfrak{S}$, let $d_S := \dim S$. If $(\mathbb{N}_S, \mathbb{L}_S)$ is a pair consisting of a normal slice and complex link, respectively, to the stratum S , then, for each integer k , the isomorphism-type of the module $\mathbb{H}^{k-d_S}(\mathbb{N}_S, \mathbb{L}_S; \mathbf{F}^\bullet)$ is independent of the choice of $(\mathbb{N}_S, \mathbb{L}_S)$; we refer to $\mathbb{H}^{k-d_S}(\mathbb{N}_S, \mathbb{L}_S; \mathbf{F}^\bullet)$ as the **degree k Morse module of S with respect to \mathbf{F}^\bullet** .

The **graded, enriched characteristic cycle of \mathbf{F}^\bullet in the cotangent bundle $T^*\mathcal{U}$** is defined in degree k to be

$$\text{gecc}^k(\mathbf{F}^\bullet) := \sum_{S \in \mathfrak{S}} H^{k-d_S}(\mathbb{N}_S, \mathbb{L}_S; \mathbf{F}^\bullet)[\overline{T_S^*\mathcal{U}}].$$

Remark 2.5. There are no canonical choices for defining the normal slices or complex links of strata. However, as two enriched cycles are equal provided that the component modules are all isomorphic, the graded, enriched characteristic cycle is well-defined.

Example 2.6. We wish to give a simple example of calculating a graded, enriched characteristic cycle.

Let $f : \mathbb{C}^3 \rightarrow \mathbb{C}$ be given by $f(x, y, t) = y(y^2 - x^3 - t^2x^2)$, and let $X := V(f) = V(y) \cup V(y^2 - x^3 - t^2x^2)$. The singular set of X , ΣX , is the 1-dimensional set $V(x, y) \cup V(x + t^2, y)$. Thus, near the origin (actually, in this specific example, globally),

$$\mathfrak{S} := \{V(y) - V(y^2 - x^3 - t^2x^2), V(y^2 - x^3 - t^2x^2) - V(y), V(x, y) - \{\mathbf{0}\}, V(x + t^2, y) - \{\mathbf{0}\}, \{\mathbf{0}\}\}$$

is a Whitney stratification of X with connected strata. Let $\mathbf{F}^\bullet := \mathbb{Z}_X^\bullet[2]$ (we shall discuss the shift by 2 below), which is constructible with respect to any Whitney stratification of X . We wish to calculate $\text{gecc}^\bullet(\mathbf{F}^\bullet)$.

First, consider the 2-dimensional strata. Let $S_1 := V(y) - V(y^2 - x^3 - t^2x^2)$. Then, \mathbb{N}_{S_1} is simply a point, and \mathbb{L}_{S_1} is empty. Hence, $H^{k-2}(\mathbb{N}_{S_1}, \mathbb{L}_{S_1}; \mathbf{F}^\bullet) = H^k(\mathbb{N}_{S_1}, \mathbb{L}_{S_1}; \mathbb{Z})$ isomorphic to \mathbb{Z} if $k = 0$, and is 0 if $k \neq 0$. The same conclusion holds if S_1 is replaced by $S_2 := V(y^2 - x^3 - t^2x^2) - V(y)$.

Now, consider the 1-dimensional strata. Let $S_3 := V(x, y) - \{\mathbf{0}\}$, and $S_4 := V(x + t^2, y) - \{\mathbf{0}\}$. The normal slice \mathbb{N}_{S_3} is, as a germ, up to analytic isomorphism, three complex lines in \mathbb{C}^2 , which intersect at a point, and \mathbb{L}_{S_3} is three points. Similarly, the normal slice \mathbb{N}_{S_4} is, as a germ, up to analytic isomorphism, two complex lines in \mathbb{C}^2 , which intersect at a point, and \mathbb{L}_{S_4} is two points. Hence, $H^{k-1}(\mathbb{N}_{S_3}, \mathbb{L}_{S_3}; \mathbf{F}^\bullet) = H^{k+1}(\mathbb{N}_{S_3}, \mathbb{L}_{S_3}; \mathbb{Z})$ isomorphic to \mathbb{Z}^2 if $k = 0$, and is 0 if $k \neq 0$. Similarly, $H^{k-1}(\mathbb{N}_{S_4}, \mathbb{L}_{S_4}; \mathbf{F}^\bullet) = H^{k+1}(\mathbb{N}_{S_4}, \mathbb{L}_{S_4}; \mathbb{Z})$ isomorphic to \mathbb{Z} if $k = 0$, and is 0 if $k \neq 0$.

Finally, consider the stratum $\{\mathbf{0}\}$. Then, $\mathbb{N}_{\{\mathbf{0}\}}$ is all of X , intersected with a small ball around the origin. The complex link $\mathbb{L}_{\{\mathbf{0}\}}$ is usually referred to as simply the complex link of X at $\mathbf{0}$. Thus, $\mathbb{L}_{\{\mathbf{0}\}}$ has the homotopy-type of a bouquet of 1-spheres (see [10]), and the number of spheres in this bouquet is equal to the intersection number $(\Gamma_{f,L}^1 \cdot V(L))_{\mathbf{0}}$, where L is any linear form such that $d_{\mathbf{0}}L$ is not a degenerate covector from strata of X at 0 (see [3]), and the relative polar curve here is the classical one from the beginning of Section 1. We claim that we may use $L := t$ for this calculation.

To see this, first note that $V(y^2 - x^3 - t^2x^2)$ is the classic example of a space such that the regular part satisfies Whitney's condition (a) along the t -axis (or, alternatively, this is an easy exercise). Thus, $d_{\mathbf{0}}t$ is not a limit of conormals from S_2 . Now, the closures of S_1 , S_3 , and S_4 are all smooth, and $d_{\mathbf{0}}t$ is not conormal to these closures at the origin.

To find the ordinary cycle $\Gamma_{f,t}^1$, we take the components of the cycle below which are **not** contained in Σf :

$$\begin{aligned} V\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) &= V(y(-3x^2 - 2t^2x), 3y^2 - x^3 - t^2x^2) = V(y, x^2(x + t^2)) + V(x(3x + 2t^2), 3y^2 - x^3 - t^2x^2) = \\ &= 2V(x, y) + V(x + t^2, y) + 2V(x, y) + V(3x + 2t^2, 3y^2 - x^3 - t^2x^2). \end{aligned}$$

Thus, $\Gamma_{f,t}^1 = V(3x + 2t^2, 3y^2 - x^3 - t^2x^2)$, and $(\Gamma_{f,t}^1 \cdot V(t))_{\mathbf{0}} = [V(3x + 2t^2, 3y^2 - x^3 - t^2x^2, t)]_{\mathbf{0}} = 2$, and $H^{k-0}(\mathbb{N}_{\{\mathbf{0}\}}, \mathbb{L}_{\{\mathbf{0}\}}; \mathbf{F}^\bullet) = H^{k+2}(\mathbb{N}_{\{\mathbf{0}\}}, \mathbb{L}_{\{\mathbf{0}\}}; \mathbb{Z})$ is isomorphic to \mathbb{Z}^2 if $k = 0$, and is 0 if $k \neq 0$.

Therefore, we find that $\text{gecc}^k(\mathbf{F}^\bullet) = 0$ if $k \neq 0$, and

$$\text{gecc}^0(\mathbf{F}^\bullet) = \mathbb{Z} \left[\overline{T_{S_1}^* \mathbb{C}^3} \right] + \mathbb{Z} \left[\overline{T_{S_2}^* \mathbb{C}^3} \right] + \mathbb{Z}^2 \left[\overline{T_{S_3}^* \mathbb{C}^3} \right] + \mathbb{Z} \left[\overline{T_{S_4}^* \mathbb{C}^3} \right] + \mathbb{Z}^2 \left[T_{\{\mathbf{0}\}}^* \mathbb{C}^3 \right].$$

The fact that $\text{gecc}^\bullet(\mathbf{F}^\bullet)$ is concentrated in degree 0 is equivalent to the fact that $\mathbb{Z}_X^\bullet[2]$ is a perverse sheaf (see [18]), and was the reason for including the shift by 2. The constant sheaf on any connected, local complete intersection, shifted by the dimension of the space, is perverse.

The reader is invited to take the most simple space Y which is not a local complete intersection – two planes P_1 and P_2 in \mathbb{C}^4 , which intersect at only the origin – and show that, if $\mathbf{A}^\bullet = \mathbb{Z}_Y^\bullet[2]$, then

$$\begin{aligned} \text{gecc}^0(\mathbf{A}^\bullet) &= \mathbb{Z} [T_{P_1}^* \mathbb{C}^4] + \mathbb{Z} [T_{P_2}^* \mathbb{C}^4], \\ \text{gecc}^{-1}(\mathbf{A}^\bullet) &= \mathbb{Z} [T_{\{\mathbf{0}\}}^* \mathbb{C}^4], \end{aligned}$$

and $\text{gecc}^k(\mathbf{A}^\bullet) = 0$ for $k \neq 0, -1$.

3 The Main Definitions and First Main Theorem

Throughout the remainder of this paper, we will use the notation established in Section 1: \mathcal{U} is an open neighborhood of the origin in \mathbb{C}^{n+1} , \tilde{f} and \tilde{g} are analytic functions from $(\mathcal{U}, \mathbf{0})$ to $(\mathbb{C}, 0)$, X is a complex analytic subset of \mathcal{U} , f and g denote the restrictions of \tilde{f} and \tilde{g} , respectively, to X , and \mathfrak{S} is a Whitney stratification of X , with connected strata, such that $V(f)$ is a union of strata.

We use (z_0, \dots, z_n) for coordinates on \mathcal{U} , and identify $T^*\mathcal{U}$ with $\mathcal{U} \times \mathbb{C}^{n+1}$, using (w_0, \dots, w_n) for cotangent coordinates, so that $(p, w_0 d_p z_0 + \dots + w_n d_p z_n)$ corresponds to $(p, (w_0, \dots, w_n))$. Let $\pi : T^*\mathcal{U} \rightarrow \mathcal{U}$ denote the projection. Below, we consider the image, $\text{im } d\tilde{g}$, of $d\tilde{g}$ in $T^*\mathcal{U}$; this scheme is defined by

$$V\left(w_0 - \frac{\partial \tilde{g}}{\partial z_0}, \dots, w_n - \frac{\partial \tilde{g}}{\partial z_n}\right) \subseteq \mathcal{U} \times \mathbb{C}^{n+1}.$$

We will consider $\text{im } d\tilde{g}$ as a scheme, an analytic set, an ordinary cycle, and as a graded, enriched cycle; we will denote all of these by simply $\text{im } d\tilde{g}$, and explicitly state what structure we are using or let the context make the structure clear.

We do not require our base ring to be \mathbb{Z} (as we did in Section 1). We let R , our base ring, be any regular, Noetherian ring with finite Krull dimension (e.g., \mathbb{Z} , \mathbb{Q} , or \mathbb{C}). This implies that every finitely-generated R -module has finite projective dimension (in fact, it implies that the projective dimension of the module is at most $\dim R$). We let \mathbf{F}^\bullet be a bounded, constructible complex of sheaves of R -modules on X . Let $\mathfrak{S}(\mathbf{F}^\bullet) := \{S \in \mathfrak{S} \mid \mathbb{H}^*(\mathbb{N}_S, \mathbb{L}_S; \mathbf{F}^\bullet) \neq 0\}$; we refer to the elements of $\mathfrak{S}(\mathbf{F}^\bullet)$ as the \mathbf{F}^\bullet -visible strata of \mathfrak{S} .

Suppose that M is a complex submanifold of \mathcal{U} . Recall:

Definition 3.1. *The relative conormal space $T_{\tilde{f}|_M}^* \mathcal{U}$ is given by*

$$T_{\tilde{f}|_M}^* \mathcal{U} := \{(x, \eta) \in T^*\mathcal{U} \mid \eta(T_x M \cap \ker d_x \tilde{f}) = 0\}.$$

If $M \subseteq X$, then $T_{\tilde{f}|_M}^ \mathcal{U}$ depends on f , but not on the particular extension \tilde{f} . In this case, we write $T_{f|_M}^* \mathcal{U}$ in place of $T_{\tilde{f}|_M}^* \mathcal{U}$.*

Definition 3.2. *The graded, enriched relative conormal cycle, $(T_{f, \mathbf{F}^\bullet}^* \mathcal{U})^\bullet$, of f , with respect to \mathbf{F}^\bullet , is defined by*

$$(T_{f, \mathbf{F}^\bullet}^* \mathcal{U})^k := \sum_{\substack{S \in \mathfrak{S}(\mathbf{F}^\bullet) \\ f|_S \neq \text{const.}}} H^{k-d_S}(\mathbb{N}_S, \mathbb{L}_S; \mathbf{F}^\bullet) \left[\overline{T_{f|_S}^* \mathcal{U}} \right].$$

Example 3.3. Let us return to the setting of Example 2.6, where $X = V(y) \cup V(y^2 - x^3 - t^2 x^2)$ and $\mathbf{F}^\bullet = \mathbb{Z}_X^\bullet[2]$. We had Whitney strata consisting of $\{\mathbf{0}\}$, $S_1 = V(y) - V(y^2 - x^3 - t^2 x^2)$, $S_2 = V(y^2 - x^3 - t^2 x^2) - V(y)$, $S_3 = V(x, y) - \{\mathbf{0}\}$, and $S_4 = V(x + t^2, y) - \{\mathbf{0}\}$.

We found that $\text{gecc}^k(\mathbf{F}^\bullet) = 0$ if $k \neq 0$, and

$$\text{gecc}^0(\mathbf{F}^\bullet) = \mathbb{Z} \left[\overline{T_{S_1}^* \mathbb{C}^3} \right] + \mathbb{Z} \left[\overline{T_{S_2}^* \mathbb{C}^3} \right] + \mathbb{Z}^2 \left[\overline{T_{S_3}^* \mathbb{C}^3} \right] + \mathbb{Z} \left[\overline{T_{S_4}^* \mathbb{C}^3} \right] + \mathbb{Z}^2 \left[T_{\{\mathbf{0}\}}^* \mathbb{C}^3 \right].$$

We will calculate $(T_{x, \mathbf{F}^\bullet}^* \mathbb{C}^3)^\bullet$.

As we said above, we identify $T^*\mathbb{C}^3$ with $\mathbb{C}^3 \times \mathbb{C}^3$, and will use coordinates (w_0, w_1, w_2) for cotangent coordinates, so that (w_0, w_1, w_2) represents $w_0 dx + w_1 dy + w_2 dt$.

Since x is identically zero on $\{0\}$ and S_3 , these two strata are not used in the calculation of $(T_{x,\mathbf{F}\bullet}^* \mathbb{C}^3)^\bullet$. For the 1-dimensional stratum S_4 , $\overline{T_{x|_{S_4}}^* \mathbb{C}^3}$ is the 4-dimensional cycle $V(x + t^2, y) \subseteq \mathbb{C}^3 \times \mathbb{C}^3$.

The fiber of $T_{x|_{S_1}}^* \mathbb{C}^3$ over any $p \in S_1$ is

$$(T_{S_1}^* \mathbb{C}^3)_p + \langle d_p x \rangle := \{\omega + ad_p x \mid \omega \in (T_{S_1}^* \mathbb{C}^3)_p, a \in \mathbb{C}\} = \{bd_p y + ad_p x \mid a, b \in \mathbb{C}\}.$$

Hence, $\overline{T_{x|_{S_1}}^* \mathbb{C}^3} = V(y, w_2)$.

The fiber of $T_{x|_{S_2}}^* \mathbb{C}^3$ over any $p \in S_2$ which is a regular point of x restricted to S_2 is

$$(T_{S_2}^* \mathbb{C}^3)_p + \langle d_p x \rangle :=$$

$$\{\omega + ad_p x \mid \omega \in (T_{S_2}^* \mathbb{C}^3)_p, a \in \mathbb{C}\} = \{b((-3x^2 - 2t^2x)d_p x + 2yd_p y - 2tx^2 d_p t) + ad_p x \mid a, b \in \mathbb{C}\}.$$

The form $w_0 d_p x + w_1 d_p y + w_2 d_p t$ is in this set if and only if the determinant of the following matrix is 0:

$$\begin{bmatrix} w_0 & w_1 & w_2 \\ -3x^2 - 2t^2x & 2y & -2tx^2 \\ 1 & 0 & 0 \end{bmatrix},$$

i.e., if and only if $yw_2 + tx^2w_1 = 0$. It is tempting to conclude that $\overline{T_{x|_{S_2}}^* \mathbb{C}^3}$ equals $V(y^2 - x^3 - t^2x^2, yw_2 + tx^2w_1)$, but this is not the case; we must eliminate any components of $V(y^2 - x^3 - t^2x^2, yw_2 + tx^2w_1)$ which are contained in $V(x, y)$. Our notation for the resulting scheme (a gap sheaf, see [17], I.1) is

$$V(y^2 - x^3 - t^2x^2, yw_2 + tx^2w_1) \dashv V(x, y).$$

Note that, as schemes,

$$\begin{aligned} V(y^2 - x^3 - t^2x^2, yw_2 + tx^2w_1) &= V(y^2 - x^3 - t^2x^2, yw_2 + tx^2w_1, y^2w_2 + ytx^2w_1) = \\ &= V(y^2 - x^3 - t^2x^2, yw_2 + tx^2w_1, (x^3 + t^2x^2)w_2 + ytx^2w_1). \end{aligned}$$

Using [17], I.1.3.iv, we find that, as cycles,

$$V(y^2 - x^3 - t^2x^2, yw_2 + tx^2w_1) \dashv V(x, y) = V(y^2 - x^3 - t^2x^2, yw_2 + tx^2w_1, (x + t^2)w_2 + ytw_1).$$

(This last equality need not be true on the level of schemes, since our generators do not form a regular sequence and, hence, there may be embedded subvarieties.)

Therefore, we find that $(T_{x,\mathbf{F}\bullet}^* \mathbb{C}^3)^k$ is 0 unless $k = 0$, and

$$(T_{x,\mathbf{F}\bullet}^* \mathbb{C}^3)^0 = \mathbb{Z}[V(y, w_2)] + \mathbb{Z}[V(y^2 - x^3 - t^2x^2, yw_2 + tx^2w_1, (x + t^2)w_2 + ytw_1)] + \mathbb{Z}[V(x + t^2, y)].$$

We now wish to define the graded, enriched relative polar curve. Note that the projection π induces an isomorphism from the analytic set $\text{im } d\tilde{g}$ to \mathcal{U} . We will use the proper push-forward (Definition 2.2) of the map π restricted to $\text{im } d\tilde{g}$; we will continue to denote this restriction by simply π .

By our conventions in Section 2, the graded, enriched $\text{im } d\tilde{g}$ is zero outside of degree 0, and is $R[\text{im } d\tilde{g}]$ in degree 0.

Definition 3.4. If $S \in \mathfrak{S}$ and $f|_S$ is not constant, we define the **relative polar set**, $|\Gamma_{f,\tilde{g}}(S)|$, to be $\pi\left(\overline{T_{f|_S}^* \mathcal{U}} \cap \text{im } d\tilde{g}\right)$; if this set is purely 1-dimensional, so that $\overline{T_{f|_S}^* \mathcal{U}}$ and $\text{im } d\tilde{g}$ intersect properly, we define the (ordinary) **relative polar cycle**, $\Gamma_{f,\tilde{g}}(S)$, to be the cycle $\pi_*\left(\left[\overline{T_{f|_S}^* \mathcal{U}}\right] \cdot [\text{im } d\tilde{g}]\right)$.

The **relative polar set**, $|\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)|$, is defined by

$$|\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)| := \pi\left(|(T_{f,\mathbf{F}^\bullet}^* \mathcal{U})^\bullet| \cap \text{im } d\tilde{g}\right).$$

Each 1-dimensional component C of $|\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)|$ is the image of a component of $|(T_{f,\mathbf{F}^\bullet}^* \mathcal{U})^\bullet| \cap \text{im } d\tilde{g}$ along which $|(T_{f,\mathbf{F}^\bullet}^* \mathcal{U})^\bullet|$ and $\text{im } d\tilde{g}$ intersect properly. We give such a component C the structure of the graded, enriched cycle whose underlying set is C and whose graded, enriched cycle structure is given by $\pi_*\left((T_{f,\mathbf{F}^\bullet}^* \mathcal{U})^\bullet \odot \text{im } d\tilde{g}\right)$ over generic points in C . We refer to this as the **graded, enriched cycle structure of C in $|\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)|$** .

If $|\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)|$ is purely 1-dimensional, we say that the **graded, enriched relative polar curve**, $(\Gamma_{f,\tilde{g}}^1(\mathbf{F}^\bullet))^\bullet$, is defined, and is given by

$$(\Gamma_{f,\tilde{g}}^1(\mathbf{F}^\bullet))^\bullet := \pi_*\left((T_{f,\mathbf{F}^\bullet}^* \mathcal{U})^\bullet \odot \text{im } d\tilde{g}\right),$$

i.e.,

$$(\Gamma_{f,\tilde{g}}^1(\mathbf{F}^\bullet))^k = \sum_{\substack{S \in \mathfrak{S}(\mathbf{F}^\bullet) \\ f|_S \neq \text{const.}}} H^{k-d_S}(\mathbb{N}_S, \mathbb{L}_S; \mathbf{F}^\bullet) (\Gamma_{f,\tilde{g}}(S))^{\text{enr}}.$$

Remark 3.5. In the notation for the polar curve, we write \tilde{g} , not simply g ; we do not, in fact, know if $(\Gamma_{f,\tilde{g}}^1(\mathbf{F}^\bullet))^\bullet$ is independent of the extension to \tilde{g} . However, Theorem 3.12 will imply that, when $(\Gamma_{f,\tilde{g}}^1(\mathbf{F}^\bullet))^\bullet$ is defined and has no component on which f is constant, then $(\Gamma_{f,\tilde{g}}^1(\mathbf{F}^\bullet))^\bullet$ is independent of the extension \tilde{g} . It is also not difficult to show that the set $|\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)|$ is independent of the extension of g , but we shall not need this result here.

Note that $\overline{T_{f|_S}^* \mathcal{U}} \cap \text{im } d\tilde{g}$ is at least 1-dimensional at each point of intersection, and so $|\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)|$ has no isolated points. Also, note that, as $|(T_{f,\mathbf{F}^\bullet}^* \mathcal{U})^\bullet| \cap \text{im } d\tilde{g}$ is a closed subset of $\text{im } d\tilde{g}$, and π induces an isomorphism from $\text{im } d\tilde{g}$ to \mathcal{U} , $|\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)|$ is a closed subset of \mathcal{U} .

Finally, the reader may wonder about the symmetry of our definition. It is **not** true for arbitrary \tilde{f} and \tilde{g} that even the sets $|\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)|$ and $|\Gamma_{g,\tilde{f}}(\mathbf{F}^\bullet)|$ are equal; see Remark 4.11. However, Proposition 4.10 will imply that the components of these two sets along which neither f nor g are constant are the same. Hence, we refer to a component of $|\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)|$ along which neither f nor g is constant as a *symmetric component* of $|\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)|$.

By moving to a generic point p on a 1-dimensional symmetric component C of $|\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)|$ and applying Corollary 4.18, one can show that the graded, enriched cycle structure of C in $|\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)|$ is the same as that of C in $|\Gamma_{g,\tilde{f}}(\mathbf{F}^\bullet)|$.

Example 3.6. We continue with our setting from Example 2.6 and Example 3.3, and consider $X = V(y) \cup V(y^2 - x^3 - t^2x^2)$ and $\mathbf{F}^\bullet = \mathbb{Z}_X^\bullet[2]$. We will calculate $(\Gamma_{x,t}^1(\mathbf{F}^\bullet))^\bullet$.

Using the isomorphism $T^*\mathbb{C}^3 \cong \mathbb{C}^3 \times \mathbb{C}^3$ from Example 3.3, $\text{im } dt$ is the scheme

$$V\left(w_0 - \frac{\partial t}{\partial x}, w_1 - \frac{\partial t}{\partial y}, w_2 - \frac{\partial t}{\partial t}\right) = V(w_0, w_1, w_2 - 1).$$

In Example 3.3, we found that $(T_{x, \mathbf{F}^\bullet}^* \mathbb{C}^3)^k$ is 0 unless $k = 0$, and

$$(T_{x, \mathbf{F}^\bullet}^* \mathbb{C}^3)^0 = \mathbb{Z}[V(y, w_2)] + \mathbb{Z}[V(y^2 - x^3 - t^2x^2, yw_2 + tx^2w_1, (x + t^2)w_2 + ytw_1)] + \mathbb{Z}[V(x + t^2, y)].$$

Let us write E for the cycle $V(y^2 - x^3 - t^2x^2, yw_2 + tx^2w_1, (x + t^2)w_2 + ytw_1)$ throughout the remainder of this example.

Thus, $(\Gamma_{x,t}^1(\mathbf{F}^\bullet))^k$ is 0 unless $k = 0$ and, to calculate $(\Gamma_{x,t}^1(\mathbf{F}^\bullet))^0$, we need first to calculate the three ordinary cycles

$$\begin{aligned} \pi_*(V(y, w_2) \cdot V(w_0, w_1, w_2 - 1)), \\ \pi_*(E \cdot V(w_0, w_1, w_2 - 1)), \end{aligned}$$

and

$$\pi_*(V(x + t^2, y) \cdot V(w_0, w_1, w_2 - 1)).$$

Now, $V(y, w_2) \cap V(w_0, w_1, w_2 - 1) = \emptyset$, and so $\pi_*(V(y, w_2) \cdot V(w_0, w_1, w_2 - 1)) = 0$. In addition, it is trivial that there is an equality of cycles $\pi_*(V(x + t^2, y) \cdot V(w_0, w_1, w_2 - 1)) = V(x + t^2, y)$. However, the remaining cycle is more difficult to calculate.

The difficulty in calculating

$$\pi_*(E \cdot V(w_0, w_1, w_2 - 1))$$

is due to the fact that $y^2 - x^3 - t^2x^2, yw_2 + tx^2w_1, (x + t^2)w_2 + ytw_1$ is not a regular sequence. To “fix” this, note that, in Example 3.3, we saw that, as cycles, there is an equality

$$V(y^2 - x^3 - t^2x^2, yw_2 + tx^2w_1) = C + E,$$

where the underlying set $|C| \subseteq V(x, y)$. Now, it is trivial that, as sets,

$$E \cap V(w_0, w_1, w_2 - 1) = V(x + t^2, y, w_0, w_1, w_2 - 1).$$

Therefore,

$$\begin{aligned} C \cdot V(w_0, w_1, w_2 - 1) + E \cdot V(w_0, w_1, w_2 - 1) &= V(y^2 - x^3 - t^2x^2, yw_2 + tx^2w_1) \cdot V(w_0, w_1, w_2 - 1) = \\ &= V(y^2 - x^3 - t^2x^2, yw_2 + tx^2w_1, w_0, w_1, w_2 - 1) = V(x^2(x + t^2), y, w_0, w_1, w_2 - 1) \\ &= 2V(x, y, w_0, w_1, w_2 - 1) + V(x + t^2, y, w_0, w_1, w_2 - 1). \end{aligned}$$

Thus, as cycles,

$$E \cdot V(w_0, w_1, w_2 - 1) = V(x + t^2, y, w_0, w_1, w_2 - 1),$$

and so $\pi_*(E \cdot V(w_0, w_1, w_2 - 1)) = V(x + t^2, y)$.

Finally, we find that

$$(\Gamma_{x,t}^1(\mathbf{F}^\bullet))^0 = \pi_*^0 \left((T_{f, \mathbf{F}^\bullet}^* \mathcal{U})^\bullet \odot \text{im } dt \right) = \mathbb{Z}[V(x + t^2, y)] + \mathbb{Z}[V(x + t^2, y)] = \mathbb{Z}^2[V(x + t^2, y)].$$

Before we can prove our main theorem of this section, we must recall three results from [18].

Theorem 3.7. ([18], Theorem 3.3) *There is an equality of graded enriched cycles given by*

$$\text{gecc}^\bullet(\psi_f[-1]\mathbf{F}^\bullet) = (T_{f, \mathbf{F}^\bullet}^* \mathcal{U})^\bullet \odot (V(f) \times \mathbb{C}^{n+1}).$$

We state the next two theorems for complexes of sheaves on $V(f)$, since that is the case in which we shall use them.

Theorem 3.8. ([18], Theorem 3.4) *Let \mathbf{A}^\bullet be a bounded, constructible complex of sheaves of R -modules on $V(f)$. Then, $\dim_{\mathbf{0}}(\text{supp } \phi_g[-1]\mathbf{A}^\bullet) \leq 0$ if and only if $\dim_{\mathbf{0}} \pi(|\text{gecc}^\bullet(\mathbf{A}^\bullet)| \cap \text{im } d\tilde{g}) \leq 0$, and when this is the case, $\dim_{(\mathbf{0}, d_{\mathbf{0}}\tilde{g})}(|\text{gecc}^\bullet(\mathbf{A}^\bullet)| \cap \text{im } d\tilde{g}) \leq 0$ and*

$$H^k(\phi_g[-1]\mathbf{A}^\bullet)_{\mathbf{0}} \cong (\text{gecc}^k(\mathbf{A}^\bullet) \odot \text{im } d\tilde{g})_{(\mathbf{0}, d_{\mathbf{0}}\tilde{g})}.$$

Theorem 3.9. ([18], Theorem 3.5) *Let \mathbf{A}^\bullet be a bounded, constructible complex of sheaves of R -modules on $V(f)$. Then, there is an equality of sets given by*

$$\bigcup_{v \in \mathbb{C}} \text{supp } \phi_{g-v}[-1]\mathbf{A}^\bullet = \pi(|\text{gecc}^\bullet(\mathbf{A}^\bullet)| \cap \text{im } d\tilde{g}).$$

We need a lemma before we prove our first main theorem.

Lemma 3.10. *There is an equality of sets*

$$\text{supp } \phi_g[-1]\psi_f[-1]\mathbf{F}^\bullet = V(f, g) \cap |\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)|,$$

and, in a neighborhood of the origin, $V(f) \cap |\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)| \subseteq V(g)$.

Proof. The equality follows from Theorem 3.9 by letting $\mathbf{A}^\bullet = \psi_f[-1]\mathbf{F}^\bullet$, applying Theorem 3.7, and then intersecting $V(g)$ with both sides of the equation from Theorem 3.9. The containment also follows from Theorem 3.9 by letting $\mathbf{A}^\bullet = \psi_f[-1]\mathbf{F}^\bullet$, applying Theorem 3.7, and then using that, near a point p where $g(p) = 0$, $\text{supp } \phi_{g-v}[-1]\mathbf{A}^\bullet$ is empty if $v \neq 0$. \square

Remark 3.11. In much of our work, particularly in [12] and [17], we have used the notion of a *prepolar slice*. Using our current terminology, the condition that $V(\tilde{g})$ is a prepolar slice for f at $\mathbf{0}$ would be replaced with $\dim_{\mathbf{0}} V(g) \cap |\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)| \leq 0$. Note that, by Lemma 3.10, this implies that $\dim_{\mathbf{0}} V(f) \cap |\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)| \leq 0$.

We now prove our first main theorem.

Theorem 3.12. *The following are equivalent:*

1. $\dim_{\mathbf{0}} \text{supp } \phi_g[-1]\psi_f[-1]\mathbf{F}^\bullet \leq 0$;
2. $\dim_{\mathbf{0}} V(f) \cap |\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)| \leq 0$;
3. $\dim_{\mathbf{0}} V(f, g) \cap |\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)| \leq 0$;

and, when these equivalent conditions hold, $(\Gamma_{f, \tilde{g}}^1(\mathbf{F}^\bullet))^\bullet$ exists and

$$H^k(\phi_g[-1]\psi_f[-1]\mathbf{F}^\bullet)_{\mathbf{0}} \cong ((\Gamma_{f, \tilde{g}}^1(\mathbf{F}^\bullet))^k \odot V(f))_{\mathbf{0}},$$

i.e.,

$$H^{k-1}(B_\epsilon \cap f^{-1}(\xi), B_\epsilon \cap f^{-1}(\xi) \cap g^{-1}(\nu); \mathbf{F}^\bullet) \cong \bigoplus_{S \in \mathfrak{S}^\circ(\mathbf{F}^\bullet)} (\mathbb{H}^{k-d_S}(\mathbb{N}_S, \mathbb{L}_S; \mathbf{F}^\bullet))^{j_S},$$

where $0 < |\xi| \ll |\nu| \ll \epsilon \ll 1$, and $j_S = (\Gamma_{f, \tilde{g}}^1(S) \cdot V(f))_{\mathbf{0}}$.

In addition, if \mathbf{C}^\bullet is the graded enriched cycle consisting of the components of $(\Gamma_{f, \tilde{g}}^1(\mathbf{F}^\bullet))^\bullet$ which are contained in $V(g)$, and the three equivalent conditions above hold, then $\dim_{\mathbf{0}} \text{supp } \psi_f[-1]\phi_g[-1]\mathbf{F}^\bullet \leq 0$ and

$$H^k(\psi_f[-1]\phi_g[-1]\mathbf{F}^\bullet)_{\mathbf{0}} \cong (\mathbf{C}^k \odot V(f))_{\mathbf{0}}.$$

Proof. The equivalence of the conditions follows immediately from the lemma. Assume now that these conditions hold.

By Theorem 3.8,

$$H^k(\phi_g[-1]\psi_f[-1]\mathbf{F}^\bullet)_0 \cong (\text{gecc}^k(\psi_f[-1]\mathbf{F}^\bullet) \odot \text{im } d\tilde{g})_{(0, d_0\tilde{g})}.$$

Applying Theorem 3.7, we find that

$$\begin{aligned} H^k(\phi_g[-1]\psi_f[-1]\mathbf{F}^\bullet)_0 &\cong ((T_{f, \mathbf{F}^\bullet}^* \mathcal{U})^k \odot (V(f) \times \mathbb{C}^{n+1}) \odot \text{im } d\tilde{g})_{(0, d_0\tilde{g})} \cong \\ &\left(\pi_*^k \left((T_{f, \mathbf{F}^\bullet}^* \mathcal{U})^\bullet \odot (V(f) \times \mathbb{C}^{n+1}) \odot \text{im } d\tilde{g} \right) \right)_0, \end{aligned}$$

where this last isomorphism follows from the definition of the proper push-forward. By Proposition 2.3, this last quantity equals

$$(\pi_*^k((T_{f, \mathbf{F}^\bullet}^* \mathcal{U})^\bullet \odot \text{im } d\tilde{g}) \odot V(f))_0,$$

which, by definition of the graded, enriched relative polar curve is equal to $((\Gamma_{f, \tilde{g}}^1(\mathbf{F}^\bullet))^k \odot V(f))_0$. This proves the first isomorphism.

Now, $H^k(\psi_f[-1]\phi_g[-1]\mathbf{F}^\bullet)_0 \cong \mathbb{H}^{k-1}(\overset{\circ}{B}_\epsilon \cap V(g) \cap V(f-a); \phi_g[-1]\mathbf{F}^\bullet)$, for $0 < |a| \ll \epsilon \ll 1$. In a neighborhood of the origin, $V(\tilde{f}-a)$ transversely intersects all strata of any complex stratification with respect to which $\phi_g[-1]\mathbf{F}^\bullet$ and \mathbf{F}^\bullet are constructible,

$$\mathbb{H}^{k-1}(\overset{\circ}{B}_\epsilon \cap V(g) \cap V(f-a); \phi_g[-1]\mathbf{F}^\bullet) \cong \mathbb{H}^{k-1}(\overset{\circ}{B}_\epsilon \cap V(g) \cap V(f-a); \phi_g[-1](\mathbf{F}^\bullet|_{V(f-a)}))$$

and, by transversality again, this is isomorphic to

$$\mathbb{H}^k(\overset{\circ}{B}_\epsilon \cap V(g) \cap V(f-a); \phi_g[-1]\psi_{f-a}[-1]\mathbf{F}^\bullet).$$

By Lemma 3.10, $\text{supp } \phi_g[-1]\psi_{f-a}[-1]\mathbf{F}^\bullet = V(g, f-a) \cap |\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)|$, which will be 0-dimensional inside $\overset{\circ}{B}_\epsilon$, by the three equivalent hypotheses. At this point, that

$$H^k(\psi_f[-1]\phi_g[-1]\mathbf{F}^\bullet)_0 \cong (C^k \odot V(f))_0$$

follows by applying the first isomorphism at each of the isolated points in $\overset{\circ}{B}_\epsilon \cap V(g, f-a) \cap |\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)|$.

□

We would like to know, of course, that the equivalent hypotheses of Theorem 3.12 are satisfied in the classical case where f is fixed and \tilde{g} is chosen to be a generic linear form.

Proposition 3.13.

1. *There exists a non-zero linear form \mathfrak{l} such that $\mathbf{0} \notin |\Gamma_{f, \mathfrak{l}}(\mathbf{F}^\bullet)|$ if and only if for generic linear \mathfrak{l} , $\mathbf{0} \notin |\Gamma_{f, \mathfrak{l}}(\mathbf{F}^\bullet)|$.*
2. *For generic linear \mathfrak{l} , $\dim_{\mathbf{0}} V(f) \cap |\Gamma_{f, \mathfrak{l}}(\mathbf{F}^\bullet)| \leq 0$ and $\dim_{\mathbf{0}} V(\mathfrak{l}) \cap |\Gamma_{f, \mathfrak{l}}(\mathbf{F}^\bullet)| \leq 0$.*

Proof. The proof of Item 1 is standard. Suppose that there exists a non-zero linear form \mathfrak{l} such that $\mathbf{0} \notin |\Gamma_{f,\mathfrak{l}}(\mathbf{F}^\bullet)|$. Then, the projective class $[d_0\mathfrak{l}]$ is not in the fiber $\mathbb{P}(|(T_{f,\mathbf{F}^\bullet}^*\mathcal{U})^\bullet|)_{\mathbf{0}} \subseteq \mathbb{P}^n$. Thus, $\mathbb{P}(|(T_{f,\mathbf{F}^\bullet}^*\mathcal{U})^\bullet|)_{\mathbf{0}}$ is a proper analytic subset of \mathbb{P}^n . This implies Item 1.

Proof of Item 2:

If \mathbf{A}^\bullet is any bounded, constructible complex of sheaves on any complex analytic $Y \subset \mathcal{U}$ and $\mathbf{0} \in Y$, then for generic linear \mathfrak{l} on U , $\mathbf{0}$ is an isolated point in $\text{supp } \phi_{\mathfrak{l}}[-1]\mathbf{A}^\bullet$. This is well-known; see, for instance, Theorem 2.4 of [16]. Thus, for generic \mathfrak{l} , $\dim_{\mathbf{0}} \text{supp } \phi_{\mathfrak{l}}[-1]\psi_f[-1]\mathbf{F}^\bullet \leq 0$, which, by Theorem 3.12, is equivalent to $\dim_{\mathbf{0}} V(f) \cap |\Gamma_{f,\mathfrak{l}}(\mathbf{F}^\bullet)| \leq 0$.

Now, refine, if necessary, our Whitney stratification \mathfrak{S} to obtain a Whitney stratification \mathfrak{S}' which also satisfies Thom's a_f condition. For generic linear \mathfrak{l} , $V(\mathfrak{l})$ will transversely intersect all of the strata of \mathfrak{S}' in a neighborhood of the origin, except possibly at the origin itself. Fix such an \mathfrak{l} . We claim that $\dim_{\mathbf{0}} V(\mathfrak{l}) \cap |\Gamma_{f,\mathfrak{l}}(\mathbf{F}^\bullet)| \leq 0$.

Since \mathfrak{S}' is an a_f stratification, $\bigcup_{S \in \mathfrak{S}'} \overline{T_{f|_S}^* \mathcal{U}} = \bigcup_{S \in \mathfrak{S}'} T_{f|_S}^* \mathcal{U}$. Our choice of \mathfrak{l} implies that, if $p \neq \mathbf{0}$ and $p \in S \in \mathfrak{S}'$, then $d_p \mathfrak{l} \notin (T_S^* \mathcal{U})_p$. Suppose now that we have an analytic path $\alpha(t)$ such that $\alpha(0) = \mathbf{0}$ and, for $t \neq 0$, $\alpha(t) \neq \mathbf{0}$ and $\alpha(t) \in V(\mathfrak{l}) \cap |\Gamma_{f,\mathfrak{l}}(\mathbf{F}^\bullet)|$. We wish to arrive at a contradiction.

If $f(\alpha(t)) = 0$ when $|t|$ is small, then we are finished, since $\dim_{\mathbf{0}} V(f) \cap |\Gamma_{f,\mathfrak{l}}(\mathbf{F}^\bullet)| \leq 0$. So, assume that for $|t|$ small and non-zero, $f(\alpha(t)) \neq 0$.

For $|t|$ small and non-zero, $\alpha(t)$ is contained in a single stratum $S \in \mathfrak{S}'$. Near the origin, the \mathfrak{S}' -stratified critical locus is contained in $V(f)$; hence, by the assumption in the previous paragraph, for $|t|$ small and non-zero, $d_{\alpha(t)} \tilde{f} \notin (T_S^* \mathcal{U})_{\alpha(t)}$. From our definition of $\alpha(t)$, and the discussion two paragraphs above, it follows that, for $|t|$ small and non-zero, $d_{\alpha(t)} \mathfrak{l} \notin (T_S^* \mathcal{U})_{\alpha(t)}$, $\mathfrak{l}(\alpha(t)) \equiv 0$, and $d_{\alpha(t)} \mathfrak{l} \in (T_{f|_S}^* \mathcal{U})_{\alpha(t)} = (T_S^* \mathcal{U})_{\alpha(t)} + \langle d_{\alpha(t)} \tilde{f} \rangle$, where the last equality uses that $d_{\alpha(t)} \tilde{f} \notin (T_S^* \mathcal{U})_{\alpha(t)}$.

Thus, for $|t|$ small and non-zero, there exists $c_t \in \mathbb{C}$ such that

$$(\dagger). \quad d_{\alpha(t)} \mathfrak{l} + c_t d_{\alpha(t)} \tilde{f} \in (T_S^* \mathcal{U})_{\alpha(t)}$$

Evaluating at $\alpha'(t)$, and using that $\mathfrak{l}(\alpha(t)) \equiv 0$ and $\alpha'(t) \in T_{\alpha(t)} S$, we immediately conclude that $c_t (f(\alpha(t)))' \equiv 0$. However, c_t cannot be zero, for otherwise (\dagger) would imply that $d_{\alpha(t)} \mathfrak{l} \in (T_S^* \mathcal{U})_{\alpha(t)}$. Therefore, we must have that $(f(\alpha(t)))' \equiv 0$, which implies that $f(\alpha(t)) \equiv 0$, since $f(\alpha(0)) = 0$. This is a contradiction. \square

Example 3.14. We continue where we left off in Example 3.6: $X = V(y) \cup V(y^2 - x^3 - t^2 x^2)$ and $\mathbf{F}^\bullet = \mathbb{Z}_X^\bullet[2]$. We found that $(\Gamma_{x,t}^1(\mathbf{F}^\bullet))^\bullet$ was concentrated in degree 0, and

$$(\Gamma_{x,t}^1(\mathbf{F}^\bullet))^0 = \mathbb{Z}^2[V(x + t^2, y)].$$

Thus, Theorem 3.12 tells us that $H^k(\phi_t[-1]\psi_x[-1]\mathbb{Z}_X^\bullet[2])_{\mathbf{0}}$ is 0 unless $k = 0$, and

$$H^0(\phi_t[-1]\psi_x[-1]\mathbb{Z}_X^\bullet[2])_{\mathbf{0}} \cong (\mathbb{Z}^2[V(x + t^2, y)] \odot V(x))_{\mathbf{0}} = \mathbb{Z}^4.$$

4 The Derived Category Discriminant

Theorem 3.12, and its elegant, formal proof, was our motivation for defining the graded, enriched relative polar curve as we did. Of course, it would be nice to have a generalization of the result of Lê in its original form, as it appears in Theorem 1.4: a result which gives $\mathbb{H}^*(F_{f,\mathbf{0}}, F_{f|_{V(g)}}, \mathbf{0}; \mathbf{F}^\bullet)$. In fact, we could easily prove

such a result by appealing to the discriminant and Cerf diagram, if only we could push the complex \mathbf{F}^\bullet down to the discriminant in some nice way.

There is one serious technical issue involved: we must show that a suitable neighborhood of origin pushes down by (g, f) to a **complex** analytically constructible complex, a complex which is constructible with respect to a stratification which is essentially determined by the image of the enriched relative polar curve. The main problem is that, on an open neighborhood of the origin, (g, f) will not be a proper map and, if we instead use a domain with boundary on which (g, f) is proper, then the boundary causes us to leave the complex analytic setting.

This is precisely the type of problem that is addressed by the microlocal theory of Kashiwara and Schapira in [6], and we will use the micro-support of complexes of sheaves on real semianalytic sets. It will take a fair amount of preliminary work before we arrive at the desired result.

Suppose that M is a C^∞ manifold, Z is a subspace of M , and \mathbf{A}^\bullet is a bounded complex of sheaves of R -modules on Z . Then, Kashiwara and Schapira define the micro-support, $SS(\mathbf{A}^\bullet) \subseteq T^*M$, of \mathbf{A}^\bullet in 5.1.2 of [6]. Intuitively, $(p, \eta) \in SS(\mathbf{A}^\bullet)$ if and only if the local hypercohomology of Z , with coefficients in \mathbf{A}^\bullet , changes as one “moves” in the direction of η .

In our fixed complex analytic setting, where X is a complex analytic subset of \mathcal{U} and \mathbf{F}^\bullet is complex analytically constructible, the micro-support is easy to describe.

Proposition 4.1. ([15], Theorem 4.13)

$$SS(\mathbf{F}^\bullet) = |\text{gecc}^\bullet(\mathbf{F}^\bullet)| = \bigcup_{S \in \mathfrak{S}(\mathbf{F}^\bullet)} \overline{T_S^* \mathcal{U}}.$$

We need to define the critical locus of complex analytic maps relative to the complex \mathbf{F}^\bullet . If $p \in X$, we shall write $SS_p(\mathbf{F}^\bullet)$ for the fiber $\pi^{-1}(p) \cap SS(\mathbf{F}^\bullet)$.

Definition 4.2. *The \mathbf{F}^\bullet -critical locus of f , $\Sigma_{\mathbf{F}^\bullet} f$, is the set $\{p \in X \mid H^*(\phi_{f-f(p)}[-1]\mathbf{F}^\bullet)_p \neq 0\}$.*

Proposition 4.3. (Theorem 2.4 and Remark 2.5 of [16]) *The closure $\overline{\Sigma_{\mathbf{F}^\bullet} f}$ is equal to $\pi(\text{im } d\tilde{f} \cap SS(\mathbf{F}^\bullet))$ and f is constant along the components of $\overline{\Sigma_{\mathbf{F}^\bullet} f}$.*

We need to generalize $\overline{\Sigma_{\mathbf{F}^\bullet} f}$ to the case where f is a real analytic map whose codomain has dimension greater than one, and where we replace \mathbf{F}^\bullet by something more general.

We may consider $T^*\mathcal{U}$ with its complex analytic structure, as we have been up to this point, or with its real analytic structure. When it is important for us to distinguish these structures, we will write $(T^*\mathcal{U})^\mathbb{C}$ and $(T^*\mathcal{U})^\mathbb{R}$, respectively, and we remind the reader that, for $p \in \mathcal{U}$, there is an \mathbb{R} -linear isomorphism from $(T^*\mathcal{U})_p^\mathbb{C}$ to $(T^*\mathcal{U})_p^\mathbb{R}$ given by mapping η to the real part $\text{Re } \eta$ (or $2\text{Re } \eta$). If $\eta_1, \dots, \eta_k \in (T^*\mathcal{U})_p^\mathbb{C}$, this isomorphism identifies the complex span $\langle \eta_1, \dots, \eta_k \rangle^\mathbb{C}$ with the real span $\langle \text{Re } \eta_1, \text{Im } \eta_1, \dots, \text{Re } \eta_k, \text{Im } \eta_k \rangle^\mathbb{R}$. When the structure is clear from the context, or is irrelevant, we shall continue to simply write $T^*\mathcal{U}$. We point out that the zero-section of $T^*\mathcal{U}$ is the conormal space to \mathcal{U} in \mathcal{U} , i.e., $T_{\mathcal{U}}^*\mathcal{U}$.

We will projectivize the fibers of $(T^*\mathcal{U})^\mathbb{C}$ (resp., $(T^*\mathcal{U})^\mathbb{R}$), and denote this projectivization by $\mathbb{P}((T^*\mathcal{U})^\mathbb{C})$ (resp., $\mathbb{P}((T^*\mathcal{U})^\mathbb{R})$), which is isomorphic to $\mathcal{U} \times \mathbb{P}^n$ (resp., $\mathcal{U} \times \mathbb{R}\mathbb{P}^{2n+1}$). In either the complex or real case, we let $\hat{\pi}$ denote the projection from the projectivization of $T^*\mathcal{U}$ to \mathcal{U} , and if η is a non-zero element of the fiber $(T^*\mathcal{U})_p$, we denote its projective class by $[\eta]$.

A subset $E \subseteq T^*\mathcal{U}$ is \mathbb{C} -conic (resp., \mathbb{R} -conic) if $(p, \eta) \in E$ implies that, for all $a \in \mathbb{C}$ (resp., $a \in \mathbb{R}$), $(p, a\eta) \in E$. If E is any subset of $T^*\mathcal{U}$, we let $\mathbb{P}(E)$ denote the (real or complex) projectivization $\{(p, [\eta]) \mid (p, \eta) \in E - T_{\mathcal{U}}^*\mathcal{U}\}$, and let $E_p := \pi^{-1}(p) \cap E$.

We need the following easy lemmas.

Lemma 4.4. *Suppose that $E \subseteq T^*\mathcal{U}$ is closed and \mathbb{R} -conic (resp., \mathbb{C} -conic). Then, $\mathbb{P}(E)$ is closed in $\mathbb{P}((T^*\mathcal{U})^{\mathbb{R}})$ (resp., $\mathbb{P}((T^*\mathcal{U})^{\mathbb{C}})$) and $\pi(E)$ is closed in \mathcal{U} .*

Proof. We shall prove the real case. The proof over the complex numbers is the same. Throughout, we shall write simply $T^*\mathcal{U}$, in place of $(T^*\mathcal{U})^{\mathbb{R}}$.

By definition of the quotient topology on $\mathbb{P}(T^*\mathcal{U})$, $\mathbb{P}(E)$ is closed if and only if

$$E' := \{(p, \eta) \in T^*\mathcal{U} - T_{\mathcal{U}}^*\mathcal{U} \mid (p, [\eta]) \in \mathbb{P}(E)\}$$

is closed in $T^*\mathcal{U} - T_{\mathcal{U}}^*\mathcal{U}$. As E is conic, $E' = E - T_{\mathcal{U}}^*\mathcal{U}$, which is closed in $T^*\mathcal{U} - T_{\mathcal{U}}^*\mathcal{U}$, since E is closed in $T^*\mathcal{U}$. Thus, $\mathbb{P}(E)$ is closed.

Now, suppose that we have a sequence $p_i \in \pi(E)$ and $p_i \rightarrow p \in \mathcal{U}$. We need to show that $p \in \pi(E)$. Let η_i be such that $(p_i, \eta_i) \in E$. Identify $T^*\mathcal{U}$ with $\mathcal{U} \times \mathbb{R}^{2n+2}$.

If an infinite number of the η_i are zero, then, by taking a subsequence (which we continue to write as p_i), we have an infinite sequence $(p_i, 0) \in E$. Then, $(p_i, 0) \rightarrow (p, 0) \in E$, as E is closed. Thus, $p = \pi(p, 0) \in \pi(E)$.

If an infinite number of the η_i are not zero, we can take a subsequence $(p_i, \eta_i/|\eta_i|)$, which is still in E , as E is conic. Since the $\eta_i/|\eta_i|$ are contained in the unit sphere, by taking another subsequence, we may assume that $\eta_i/|\eta_i|$ converges to some η . Thus, $(p_i, \eta_i/|\eta_i|) \rightarrow (p, \eta)$, which is in E , since E is closed, and so $p \in \pi(E)$. \square

Lemma 4.5. *Suppose that $\tilde{h}_1, \dots, \tilde{h}_k$ are real (resp., complex) analytic functions from \mathcal{U} to \mathbb{R} (resp., \mathbb{C}), and suppose that $E \subseteq T^*\mathcal{U}$ is closed and \mathbb{R} -conic (resp., \mathbb{C} -conic). Then, the set $\overline{\Sigma}_E(\tilde{h}_1, \dots, \tilde{h}_k)$ of $p \in \mathcal{U}$ such that there exists non-zero $(a_1, \dots, a_k) \in \mathbb{R}^k$ (resp., \mathbb{C}^k) such that*

$$a_1 d_p \tilde{h}_1 + \dots + a_k d_p \tilde{h}_k \in E_p$$

is closed in \mathcal{U} .

Proof. We shall prove the real case. The proof over the complex numbers is the same. Let $N := 2n + 2$.

Let K be the set of points $p \in \mathcal{U}$ such that $d_p \tilde{h}_1, \dots, d_p \tilde{h}_k$ are linearly dependent, i.e., let K be the critical locus of the map $(\tilde{h}_1, \dots, \tilde{h}_k)$. Note that K is closed.

Consider the continuous function $T : (\mathcal{U} - K) \times \mathbb{R}\mathbb{P}^{k-1} \rightarrow (\mathcal{U} - K) \times \mathbb{R}\mathbb{P}^{N-1}$ given by $T(p, [a_1, \dots, a_k]) = (p, [a_1 d_p \tilde{h}_1 + \dots + a_k d_p \tilde{h}_k])$. Let $\tilde{\pi} : (\mathcal{U} - K) \times \mathbb{R}\mathbb{P}^{k-1} \rightarrow \mathcal{U} - K$ denote the projection map.

By Lemma 4.4, $\mathbb{P}(E)$ is closed in $\mathcal{U} \times \mathbb{R}\mathbb{P}^{N-1}$. Therefore, $B := T^{-1}((\mathcal{U} - K) \times \mathbb{R}\mathbb{P}^{N-1}) \cap \mathbb{P}(E)$ is closed in $(\mathcal{U} - K) \times \mathbb{R}\mathbb{P}^{k-1}$. As $\tilde{\pi}$ is proper, $\tilde{\pi}(B)$ is closed in $\mathcal{U} - K$.

Now, the set $\overline{\Sigma}_E(\tilde{h}_1, \dots, \tilde{h}_k)$ is equal to $\tilde{\pi}(B) \cup K$, which is closed in \mathcal{U} . \square

Definition 4.6. *Let $\tilde{H} := (\tilde{h}_1, \dots, \tilde{h}_k)$ and E be as in Lemma 4.5. Then, the set $\overline{\Sigma}_E \tilde{H}$ from Lemma 4.5 is the **closed E -critical locus of \tilde{H}** . We define the **E -discriminant of \tilde{H}** , $\Delta_E \tilde{H}$, to be $\tilde{H}(\overline{\Sigma}_E \tilde{H})$.*

If \tilde{H} is complex analytic, and $E = SS(\mathbf{F}^\bullet)$, then we let $\overline{\Sigma}_{\mathbf{F}^\bullet} \tilde{H} := \overline{\Sigma}_E \tilde{H}$ and $\Delta_{\mathbf{F}^\bullet} \tilde{H} := \Delta_E \tilde{H}$.

Remark 4.7. By Proposition 4.3, if $E := SS(\mathbf{F}^\bullet)$, then $\overline{\Sigma}_{\mathbf{F}^\bullet} f = \overline{\Sigma}_E f$; this was our reason for adopting our notation for the closed E -critical locus.

While we shall not need it in this paper, it is possible to show that, in special cases, there is a reasonable notion of the (non-closed) critical locus $\Sigma_E(h_1, \dots, h_k)$ which depends only on the restriction (h_1, \dots, h_k) of $(\tilde{h}_1, \dots, \tilde{h}_k)$ to $\pi(E)$, and $\overline{\Sigma}_E(h_1, \dots, h_k) = \overline{\Sigma}_E(\tilde{h}_1, \dots, \tilde{h}_k)$. In particular, this is the case when \tilde{H} is complex analytic and $E = SS(\mathbf{F}^\bullet)$.

Lemma 4.8. *Suppose that Z is an analytic subset of \mathcal{U} such that \tilde{f} is not constant on any irreducible component of Z . Let M be an open dense subset of Z_{reg} . Then, for all $p \in Z$,*

$$\overline{(T_M^* \mathcal{U})}_p + \langle d_p \tilde{f} \rangle := \{\omega + ad_p \tilde{f} \mid a \in \mathbb{C}, \omega \in \overline{(T_M^* \mathcal{U})}_p\} \subseteq \overline{(T_{\tilde{f}|_M}^* \mathcal{U})}_p,$$

and, if $d_p \tilde{f} \notin \overline{(T_M^* \mathcal{U})}_p$, then this containment is an equality.

Proof. Since \tilde{f} is not constant on any component of Z , $M' := M - \Sigma(\tilde{f}|_M)$ is dense in Z . Thus, $\overline{T_M^* \mathcal{U}} = \overline{T_{M'}^* \mathcal{U}}$, and $\overline{T_{\tilde{f}|_M}^* \mathcal{U}} = \overline{T_{\tilde{f}|_{M'}}^* \mathcal{U}}$.

Consider $\eta := \omega + ad_p \tilde{f} \in \overline{(T_M^* \mathcal{U})}_p + \langle d_p \tilde{f} \rangle$, where $(p_i, \omega_i) \in T_{M'}^* \mathcal{U}$ and $(p_i, \omega_i) \rightarrow (p, \omega)$. Then, $\eta_i := \omega_i + ad_{p_i} \tilde{f} \in \overline{(T_{\tilde{f}|_{M'}}^* \mathcal{U})}_{p_i}$, and $\eta_i \rightarrow \eta$. Therefore, the containment holds.

Suppose now that $d_p \tilde{f} \notin \overline{(T_M^* \mathcal{U})}_p$, and that $\eta \in \overline{(T_{\tilde{f}|_M}^* \mathcal{U})}_p = \overline{(T_{\tilde{f}|_{M'}}^* \mathcal{U})}_p$. Then, there exists an analytic path $(p(t), \eta_t) \in \overline{T_{\tilde{f}|_{M'}}^* \mathcal{U}}$ such that $p = p(0)$, $\eta = \eta_0$, and, for $t \neq 0$, $(p(0), \eta_t) \in T_{\tilde{f}|_{M'}}^* \mathcal{U}$. Hence, for $t \neq 0$, $\eta_t = \omega_t + \lambda(t) d_{p(t)} \tilde{f}$, where $\lambda(t) \in \mathbb{C}$ and $\omega_t \in (T_{M'}^* \mathcal{U})_{p(t)}$, and $\lambda(t)$ and ω_t are uniquely determined. Evaluating at $p'(t)$, we find that, for $t \neq 0$, $\eta_t(p'(t)) = \lambda(t) d_{p(t)} \tilde{f}(p'(t))$, and so $\lambda(t)$ is a quotient of two analytic functions. Therefore, there are two possibilities as $t \rightarrow 0$: either $\lambda(t)$ approaches some $a \in \mathbb{C}$, or $|\lambda(t)| \rightarrow \infty$.

If $\lambda(t) \rightarrow a \in \mathbb{C}$, then $\omega_t \rightarrow \eta - ad_p \tilde{f}$, and $\eta = (\eta - ad_p \tilde{f}) + ad_p \tilde{f} \in \overline{(T_M^* \mathcal{U})}_p + \langle d_p \tilde{f} \rangle$. We claim that $d_p \tilde{f} \notin \overline{(T_M^* \mathcal{U})}_p$ implies that the case $|\lambda(t)| \rightarrow \infty$ cannot occur. Once we show this, the proof will be finished.

If $|\lambda(t)| \rightarrow \infty$, then, as $\eta_t \rightarrow \eta$,

$$\frac{\omega_t}{\lambda(t)} + d_{p(t)} \tilde{f} = \frac{\eta_t}{\lambda(t)} \rightarrow 0,$$

i.e.,

$$d_p \tilde{f} = \lim_{t \rightarrow 0} \left(-\frac{\omega_t}{\lambda(t)} \right) \in \overline{(T_M^* \mathcal{U})}_p.$$

□

Lemma 4.9.

1. $SS_p(\mathbf{F}^\bullet) + \langle d_p \tilde{f} \rangle \subseteq |(T_{f, \mathbf{F}^\bullet}^* \mathcal{U})^\bullet|_p \cup SS_p(\mathbf{F}^\bullet)$.
2. Suppose that $p \notin \overline{\Sigma_{\mathbf{F}^\bullet} f}$. Then, $|(T_{f, \mathbf{F}^\bullet}^* \mathcal{U})^\bullet|_p = SS_p(\mathbf{F}^\bullet) + \langle d_p \tilde{f} \rangle$.

Proof.

Proof of Item 1:

Suppose that $\eta \in SS_p(\mathbf{F}^\bullet) + \langle d_p \tilde{f} \rangle$. Then, $\eta = \omega + ad_p \tilde{f}$, where $a \in \mathbb{C}$ and $\omega \in \overline{(T_S^* \mathcal{U})}_p$ for some $S \in \mathfrak{S}(\mathbf{F}^\bullet)$. Then, there exist $(p_i, \omega_i) \in T_S^* \mathcal{U}$ such that $(p_i, \omega_i) \rightarrow (p, \omega)$. There are two cases.

If f is constant on S , then $(p_i, \omega_i + ad_{p_i} \tilde{f}) \in T_S^* \mathcal{U}$. Hence, $\eta = \omega + ad_p \tilde{f} \in \overline{(T_S^* \mathcal{U})}_p \subseteq SS_p(\mathbf{F}^\bullet)$. If f is not constant on S , then Lemma 4.8 implies that $\eta \in |(T_{f, \mathbf{F}^\bullet}^* \mathcal{U})^\bullet|_p$.

Proof of Item 2:

Note that if $S \in \mathfrak{S}(\mathbf{F}^\bullet)$ and $f|_S$ is constant, then Proposition 4.3 implies that $S \subseteq \overline{\Sigma_{\mathbf{F}^\bullet} f}$; hence, by our hypothesis, $p \notin \overline{S}$. Therefore,

$$SS_p(\mathbf{F}^\bullet) = \pi^{-1}(p) \cap \bigcup_{\substack{S \in \mathfrak{S}(\mathbf{F}^\bullet) \\ f|_S \neq \text{const.}}} \overline{T_S^* \mathcal{U}}.$$

Now the result follows immediately from Lemma 4.8 and the definition of $(T_{f, \mathbf{F}^\bullet}^* \mathcal{U})^\bullet$. \square

Proposition 4.10. *There is an equality of sets*

$$\overline{\Sigma_{\mathbf{F}^\bullet}(\tilde{f}, \tilde{g})} = \overline{\Sigma_{\mathbf{F}^\bullet} f} \cup |\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)|.$$

Proof. Suppose that $p \notin \overline{\Sigma_{\mathbf{F}^\bullet} f}$. Then, it follows immediately from Item 2 of Lemma 4.9 that $p \in \overline{\Sigma_{\mathbf{F}^\bullet}(\tilde{f}, \tilde{g})}$ if and only if $p \in |\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)|$. Therefore,

$$\overline{\Sigma_{\mathbf{F}^\bullet}(\tilde{f}, \tilde{g})} - \overline{\Sigma_{\mathbf{F}^\bullet} f} = |\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)| - \overline{\Sigma_{\mathbf{F}^\bullet} f}.$$

Now, take the union of both sides above with $\overline{\Sigma_{\mathbf{F}^\bullet} f}$, and use that $\overline{\Sigma_{\mathbf{F}^\bullet} f} \subseteq \overline{\Sigma_{\mathbf{F}^\bullet}(\tilde{f}, \tilde{g})}$. \square

Remark 4.11. Note that there is no claim in Proposition 4.10 about $\overline{\Sigma_{\mathbf{F}^\bullet} f}$ and $|\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)|$ intersecting in some nice way. In fact, in Remark 4.15, we give an example where these two sets are equal. However, if $|\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)|$ is 1-dimensional and f is not constant along any component of $|\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)|$, then Proposition 4.3 implies that the intersection of $\overline{\Sigma_{\mathbf{F}^\bullet} f}$ and $|\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)|$ is either empty or consists of isolated points.

We also want to return to the topic of symmetry that we first discussed in Remark 3.5. By Proposition 4.10 and the symmetry of the definition of the closed critical locus, we have that

$$\overline{\Sigma_{\mathbf{F}^\bullet} f} \cup |\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)| = \overline{\Sigma_{\mathbf{F}^\bullet} g} \cup |\Gamma_{g, \tilde{f}}(\mathbf{F}^\bullet)|.$$

By Proposition 4.3, f and g are constant along the components of $\overline{\Sigma_{\mathbf{F}^\bullet} f}$ and $\overline{\Sigma_{\mathbf{F}^\bullet} g}$, respectively. It follows that the symmetric components of $|\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)|$ and $|\Gamma_{g, \tilde{f}}(\mathbf{F}^\bullet)|$ are the same.

Note, however, that even in the classical case where we look at germs at the origin, f is fixed, and \tilde{g} is chosen to be a generic linear form, it is, in general, **false** that there is an equality of sets $|\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)| = |\Gamma_{g, \tilde{f}}(\mathbf{F}^\bullet)|$. Suppose, for instance, that $\dim_0 \overline{\Sigma_{\mathbf{F}^\bullet} f} \geq 2$. For a generic linear form \mathfrak{l} , either $\overline{\Sigma_{\mathbf{F}^\bullet} \mathfrak{l}}$ will be empty or the origin will be an isolated point in $\overline{\Sigma_{\mathbf{F}^\bullet} \mathfrak{l}}$; furthermore, Proposition 3.13 implies that $|\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)|$ is purely 1-dimensional at the origin. However,

$$|\Gamma_{g, \tilde{f}}(\mathbf{F}^\bullet)| = \overline{\Sigma_{\mathbf{F}^\bullet} g} \cup |\Gamma_{g, \tilde{f}}(\mathbf{F}^\bullet)| = \overline{\Sigma_{\mathbf{F}^\bullet} f} \cup |\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)|$$

is, at least, 2-dimensional at the origin.

We now need to prove our main technical lemma.

Let $\mathring{\mathbb{D}}_\delta$ denote an open disk of radius δ , centered at the origin, in \mathbb{C} . For positive $\rho, \delta, \epsilon \in \mathbb{R}$, let $N_{\delta, \rho}^\epsilon := B_\epsilon \cap g^{-1}(\mathring{\mathbb{D}}_\delta) \cap f^{-1}(\mathring{\mathbb{D}}_\rho)$, let $P_{\delta, \rho}^\epsilon$ be the restriction of (\tilde{g}, \tilde{f}) to a map from $N_{\delta, \rho}^\epsilon$ to $\mathring{\mathbb{D}}_\delta \times \mathring{\mathbb{D}}_\rho$, and let $(\mathbf{F}^\bullet)_{\delta, \rho}^\epsilon$ denote the restriction of \mathbf{F}^\bullet to $N_{\delta, \rho}^\epsilon$. Let $r : \mathcal{U} \rightarrow \mathbb{R}$ be the ‘‘squared distance from the origin’’ function $r = |z_0|^2 + \dots + |z_n|^2$.

Lemma 4.12. *Suppose that $\dim_0 V(f) \cap |\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)| \leq 0$. Then, there exists $\epsilon_0 > 0$ such that, for all ϵ_1 and ϵ_2 such that $0 < \epsilon_2 < \epsilon_1 \leq \epsilon_0$, there exist $\delta, \rho > 0$ such that*

1. for all $p \in N_{\delta, \rho}^{\epsilon_1} - \mathring{B}_{\epsilon_2}$,

$$d_p r \notin (SS_p(\mathbf{F}^\bullet) + \langle d_p \tilde{g}, d_p \tilde{f} \rangle)^\mathbb{R};$$

2. for all $p \in (N_{\delta, \rho}^{\epsilon_1} - \overset{\circ}{B}_{\epsilon_2}) - \overline{\Sigma_{\mathbf{F}^\bullet} f}$, in particular, for all $p \in (N_{\delta, \rho}^{\epsilon_1} - \overset{\circ}{B}_{\epsilon_2}) - V(f)$, for all $(a, b) \in \mathbb{C}^2 - \{\mathbf{0}\}$,

$$\operatorname{Re}(ad_p \tilde{f} + bd_p \tilde{g}) \notin (SS_p(\mathbf{F}^\bullet))^{\mathbb{R}} + \langle d_p r \rangle^{\mathbb{R}}.$$

Proof.

Proof of Item 1:

Let Y be the set of $p \in U$ such that there exists non-zero $(c, b) \in \mathbb{R} \times \mathbb{C}$ such that $cd_p r + \operatorname{Re}(bd_p \tilde{g}) \in |(T_{f, \mathbf{F}^\bullet}^* \mathcal{U})^\bullet|_p^{\mathbb{R}}$. By Lemma 4.5, Y is closed in \mathcal{U} . Let Z be the set of $p \in U$ such that

$$d_p r \in (SS_p(\mathbf{F}^\bullet) + \langle d_p \tilde{g}, d_p \tilde{f} \rangle)^{\mathbb{R}}.$$

Let $B_{\epsilon_0}^* := B_{\epsilon_0} - \{\mathbf{0}\}$.

We shall prove Item 1 by proving that there exists $\epsilon_0 > 0$ such that

- a. $B_{\epsilon_0}^* \cap Y \cap V(f, g) = \emptyset$;
- b. $B_{\epsilon_0}^* \cap Z - V(f) \subseteq B_{\epsilon_0}^* \cap Y - V(f)$; and
- c. for all ϵ_1, ϵ_2 such that $0 < \epsilon_2 < \epsilon_1 \leq \epsilon_0$, there exists $\delta' > 0$ such that

$$(B_{\epsilon_1} - \overset{\circ}{B}_{\epsilon_2}) \cap Z \cap g^{-1}(\overset{\circ}{\mathbb{D}}_{\delta'}) \cap V(f) = \emptyset.$$

We will first show how Items a, b, and c imply Item 1. We will then show that Items a, b, and c hold.

Assume Items a, b, and c, and let ϵ_1, ϵ_2 be such that $0 < \epsilon_2 < \epsilon_1 \leq \epsilon_0$. Let δ' be as in Item c. As Y is closed, $(B_{\epsilon_1} - \overset{\circ}{B}_{\epsilon_2}) \cap Y$ is compact. Thus, since $(B_{\epsilon_1} - \overset{\circ}{B}_{\epsilon_2}) \cap Y \cap V(f, g) = \emptyset$ by Item a, there exist $\rho, \delta'' > 0$ such that

$$(\dagger) \quad (B_{\epsilon_1} - \overset{\circ}{B}_{\epsilon_2}) \cap Y \cap g^{-1}(\overset{\circ}{\mathbb{D}}_{\delta''}) \cap f^{-1}(\overset{\circ}{\mathbb{D}}_{\rho}) = \emptyset.$$

Fix such ρ and δ'' . Fix δ such that $0 < \delta \leq \min\{\delta', \delta''\}$. We wish to show that $(N_{\delta, \rho}^{\epsilon_1} - \overset{\circ}{B}_{\epsilon_2}) \cap Z = \emptyset$. Suppose that $p \in (N_{\delta, \rho}^{\epsilon_1} - \overset{\circ}{B}_{\epsilon_2}) \cap Z$. Then, $p \notin V(f)$ by Item c. However, then, Item b implies that $p \in Y$; a contradiction of (\dagger) .

Now we will show that we may pick $\epsilon_0 > 0$ so that Items a, b, and c hold. Choose $\epsilon_0 > 0$ such that, for all ϵ such that $0 < \epsilon \leq \epsilon_0$, ∂B_ϵ transversely intersects all of the strata of \mathfrak{S} . Then, for all $p \in B_{\epsilon_0} - \{\mathbf{0}\}$, $d_p r \notin (SS_p(\mathbf{F}^\bullet))^{\mathbb{R}}$. Similarly, we may also choose $\epsilon_0 > 0$ so that, for all ϵ such that $0 < \epsilon \leq \epsilon_0$, for all $p \in (B_{\epsilon_0} - \{\mathbf{0}\})$,

$$d_p r \notin (SS_p(\psi_f[-1]\mathbf{F}^\bullet))^{\mathbb{R}} \cup (SS_p(\psi_g[-1]\mathbf{F}^\bullet))^{\mathbb{R}} \cup (SS_p(\psi_g[-1]\psi_f[-1]\mathbf{F}^\bullet))^{\mathbb{R}}.$$

Combining the paragraph above with Proposition 4.3 and the equivalences at the beginning of Theorem 3.12, and using our hypothesis that $\dim_{\mathbf{0}} V(f) \cap |\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)| \leq 0$, we may pick $\epsilon_0 > 0$ such that

- i. $B_{\epsilon_0}^* \cap \overline{\Sigma_{\mathbf{F}^\bullet} f} \subseteq V(f)$;
- ii. $B_{\epsilon_0}^* \cap \overline{\Sigma_{\psi_f[-1]\mathbf{F}^\bullet} g} = \emptyset$;
- iii. for all $p \in B_{\epsilon_0}^*$,

$$d_p r \notin (SS_p(\mathbf{F}^\bullet))^{\mathbb{R}} \cup (SS_p(\psi_f[-1]\mathbf{F}^\bullet))^{\mathbb{R}} \cup (SS_p(\psi_g[-1]\mathbf{F}^\bullet))^{\mathbb{R}} \cup (SS_p(\psi_g[-1]\psi_f[-1]\mathbf{F}^\bullet))^{\mathbb{R}}.$$

Proof of Item a:

Suppose that we have non-zero $(c, b) \in \mathbb{R} \times \mathbb{C}$ and $p \in B_{\epsilon_0}^* \cap V(f, g)$ such that $cd_p r + \operatorname{Re}(bd_p \tilde{g}) \in |(T_{f, \mathbf{F}^\bullet}^* \mathcal{U})^\bullet|_p^\mathbb{R} = (SS_p(\psi_f[-1]\mathbf{F}^\bullet))^\mathbb{R}$, where the last equality follows from Theorem 3.7. If $c = 0$, then $d_p \tilde{g} \in SS_p(\psi_f[-1]\mathbf{F}^\bullet)$, i.e., $p \in \overline{\Sigma}_{\psi_f[-1]\mathbf{F}^\bullet} g$, which contradicts Item ii. Thus, c must be unequal to zero, and so $d_p r \in (SS_p(\psi_f[-1]\mathbf{F}^\bullet) + \langle d_p \tilde{g} \rangle)^\mathbb{R}$. As $d_p \tilde{g} \notin SS_p(\psi_f[-1]\mathbf{F}^\bullet)$ and $p \in V(g)$, by Lemma 4.9, Item 2, $SS_p(\psi_f[-1]\mathbf{F}^\bullet) + \langle d_p \tilde{g} \rangle = SS_p(\psi_g[-1]\psi_f[-1]\mathbf{F}^\bullet)$. However, $d_p r \in SS_p(\psi_g[-1]\psi_f[-1]\mathbf{F}^\bullet)$ contradicts Item iii. This proves Item a.

Proof of Item b:

Suppose that $p \in B_{\epsilon_0}^* \cap Z - V(f)$, i.e., $p \in B_{\epsilon_0}^* - V(f)$ and there exists $b \in \mathbb{C}$ such that $d_p r + \operatorname{Re}(bd_p \tilde{g}) \in (SS_p(\mathbf{F}^\bullet) + \langle d_p \tilde{f} \rangle)^\mathbb{R}$. By Item i and Lemma 4.9, Item 2, $SS_p(\mathbf{F}^\bullet) + \langle d_p \tilde{f} \rangle = |(T_{f, \mathbf{F}^\bullet}^* \mathcal{U})^\bullet|_p$, and so $d_p r + \operatorname{Re}(bd_p \tilde{g}) \in |(T_{f, \mathbf{F}^\bullet}^* \mathcal{U})^\bullet|_p^\mathbb{R}$. Hence, $p \in Y$, and we have proved Item b.

Proof of Item c:

Let W be the closed set

$$W := \{p \in \mathcal{U} \mid d_p r \in (|(T_{g, \psi_f[-1]\mathbf{F}^\bullet}^* \mathcal{U})^\bullet|_p \cup |(T_{g, \mathbf{F}^\bullet}^* \mathcal{U})^\bullet|_p \cup SS_p(\psi_f[-1]\mathbf{F}^\bullet) \cup SS_p(\mathbf{F}^\bullet))^\mathbb{R}\}.$$

By Item iii and Theorem 3.7, $B_{\epsilon_0}^* \cap W \cap V(g) = \emptyset$. We also claim that

$$(\ddagger) \quad B_{\epsilon_0}^* \cap Z \cap V(f) \subseteq B_{\epsilon_0}^* \cap W \cap V(f).$$

For $p \in Z$ if and only if there exists a non-zero $b \in \mathbb{C}$ such that $d_p r + \operatorname{Re}(bd_p \tilde{g}) \in (SS_p(\mathbf{F}^\bullet) + \langle d_p \tilde{f} \rangle)^\mathbb{R}$, and, by Item 1 of Lemma 4.9,

$$SS_p(\mathbf{F}^\bullet) + \langle d_p \tilde{f} \rangle \subseteq |(T_{f, \mathbf{F}^\bullet}^* \mathcal{U})^\bullet|_p \cup SS_p(\mathbf{F}^\bullet).$$

If $p \in V(f)$, then Theorem 3.7 implies that $|(T_{f, \mathbf{F}^\bullet}^* \mathcal{U})^\bullet|_p = SS_p(\psi_f[-1]\mathbf{F}^\bullet)$. Now, (\ddagger) follows at once.

Let ϵ_1, ϵ_2 be such that $0 < \epsilon_2 < \epsilon_1 \leq \epsilon_0$. Then, since $B_{\epsilon_0}^* \cap W \cap V(g) = \emptyset$ and W is closed, $(B_{\epsilon_1} - \overset{\circ}{B}_{\epsilon_2}) \cap W$ is compact and $(B_{\epsilon_1} - \overset{\circ}{B}_{\epsilon_2}) \cap W \cap V(g) = \emptyset$. Thus, there exists $\delta' > 0$ such that $(B_{\epsilon_1} - \overset{\circ}{B}_{\epsilon_2}) \cap W \cap g^{-1}(\overset{\circ}{\mathbb{D}}_{\delta'}) = \emptyset$. Finally, we conclude that

$$(B_{\epsilon_1} - \overset{\circ}{B}_{\epsilon_2}) \cap Z \cap g^{-1}(\overset{\circ}{\mathbb{D}}_{\delta'}) \cap V(f) \subseteq (B_{\epsilon_1} - \overset{\circ}{B}_{\epsilon_2}) \cap W \cap g^{-1}(\overset{\circ}{\mathbb{D}}_{\delta'}) \cap V(f) = \emptyset,$$

which proves Item c, and concludes the proof of Item 1 from the statement of the lemma.

Proof of Item 2:

Assume that $\dim_{\mathbf{0}} V(f) \cap |\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)| \leq 0$. Pick $\epsilon_0 > 0$ so that $B_{\epsilon_0} \cap V(f) \cap |\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)| \subseteq \{\mathbf{0}\}$. This means precisely that

$$(*) \quad (B_{\epsilon_0} - \{\mathbf{0}\}) \cap \{p \in \mathcal{U} \mid d_p \tilde{g} \in |(T_{f, \mathbf{F}^\bullet}^* \mathcal{U})^\bullet|_p\} \cap V(f) = \emptyset.$$

Let ϵ_1 and ϵ_2 be such that $0 < \epsilon_2 < \epsilon_1 \leq \epsilon_0$, and assume that ρ and δ are such that Item 1 holds. Then,

$$(B_{\epsilon_1} - \overset{\circ}{B}_{\epsilon_2}) \cap V(f) \cap |\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)| = \emptyset.$$

As $(B_{\epsilon_1} - \overset{\circ}{B}_{\epsilon_2}) \cap |\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)|$ is compact, it follows that we may re-choose ρ , smaller if needed, so that

$$(\dagger) \quad (B_{\epsilon_1} - \overset{\circ}{B}_{\epsilon_2}) \cap f^{-1}(\overset{\circ}{\mathbb{D}}_{\rho}) \cap |\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)| = \emptyset.$$

We claim that Item 2 holds.

To see this, let $p \in (N_{\delta, \rho}^{\epsilon_1} - \overset{\circ}{B}_{\epsilon_2}) - \overline{\Sigma_{\mathbf{F}^\bullet} f}$, and assume that we have $(a, b) \in \mathbb{C}^2 - \{\mathbf{0}\}$ such that

$$\operatorname{Re}(ad_p \tilde{f} + bd_p \tilde{g}) \in (SS_p(\mathbf{F}^\bullet))^{\mathbb{R}} + \langle d_p r \rangle^{\mathbb{R}}.$$

Then, there exists a real number c such that

$$cd_p r + \operatorname{Re}(ad_p \tilde{f} + bd_p \tilde{g}) \in (SS_p(\mathbf{F}^\bullet))^{\mathbb{R}}.$$

If $c \neq 0$, we may divide by c and obtain a contradiction to Item 1. Thus, c must equal 0, and so

$$ad_p \tilde{f} + bd_p \tilde{g} \in SS_p(\mathbf{F}^\bullet),$$

i.e., $p \in \overline{\Sigma_{\mathbf{F}^\bullet}(\tilde{f}, \tilde{g})}$, which by Proposition 4.10, is equal to $\overline{\Sigma_{\mathbf{F}^\bullet} f} \cup |\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)|$. This is a contradiction of (\dagger) and the fact that $p \notin \overline{\Sigma_{\mathbf{F}^\bullet} f}$. \square

As before, for $\epsilon, \delta, \rho > 0$, let $N_{\delta, \rho}^\epsilon := B_\epsilon \cap g^{-1}(\overset{\circ}{\mathbb{D}}_\delta) \cap f^{-1}(\overset{\circ}{\mathbb{D}}_\rho)$. Let $(\mathbf{F}^\bullet)_{\delta, \rho}^\epsilon$ be the restriction of \mathbf{F}^\bullet to $N_{\delta, \rho}^\epsilon$, and let $T_{\delta, \rho}^\epsilon$ be the restriction of the map (g, f) to a map from $N_{\delta, \rho}^\epsilon$ to $\overset{\circ}{\mathbb{D}}_\delta \times \overset{\circ}{\mathbb{D}}_\rho$.

Theorem 4.13. (The Derived Category Discriminant Theorem) *Suppose that $\dim_{\mathbf{0}} V(f) \cap |\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)| \leq 0$.*

Then, for all sufficiently small $\epsilon > 0$, there exist $\delta, \rho > 0$ such that the (derived) push-forward $\mathbf{A}^\bullet := (T_{\delta, \rho}^\epsilon)_(\mathbf{F}^\bullet)_{\delta, \rho}^\epsilon$ is complex analytically constructible with respect to the stratification given by*

$$\{\overset{\circ}{\mathbb{D}}_\delta \times \overset{\circ}{\mathbb{D}}_\rho - \Delta_{\mathbf{F}^\bullet}(\tilde{g}, \tilde{f}), (\overset{\circ}{\mathbb{D}}_\delta \times \overset{\circ}{\mathbb{D}}_\rho) \cap \Delta_{\mathbf{F}^\bullet}(\tilde{g}, \tilde{f}) - \{\mathbf{0}\}, \{\mathbf{0}\}\}.$$

Proof. Fix choices of $\epsilon_0, \epsilon_1, \epsilon_2, \delta$, and ρ as in Lemma 4.12. Pick ϵ so that $\epsilon_2 < \epsilon < \epsilon_1$. Let \mathbf{G}^\bullet be the restriction of \mathbf{F}^\bullet to $Y := \overset{\circ}{B}_{\epsilon_1} \cap g^{-1}(\overset{\circ}{\mathbb{D}}_\delta) \cap f^{-1}(\overset{\circ}{\mathbb{D}}_\rho)$. Let \tilde{h} be the restriction of (g, f) to a map from Y to $\overset{\circ}{\mathbb{D}}_\delta \times \overset{\circ}{\mathbb{D}}_\rho$. Then, by applying Item 1 of Lemma 4.12 and Proposition 8.5.8 of [6] to \mathbf{G}^\bullet (where the ϕ and f of [6] are our r and (g, f) , respectively), we immediately conclude that \mathbf{A}^\bullet is complex analytically constructible.

As $\Delta_{\mathbf{F}^\bullet}(\tilde{g}, \tilde{f})$ is either empty or a curve, to show that \mathbf{A}^\bullet is constructible with respect to the given stratification, one has only to show that the cohomology of \mathbf{A}^\bullet is locally constant at points in $\overset{\circ}{\mathbb{D}}_\delta \times \overset{\circ}{\mathbb{D}}_\rho - \Delta_{\mathbf{F}^\bullet}(\tilde{g}, \tilde{f})$. Let $q := (u_0, v_0) \in \overset{\circ}{\mathbb{D}}_\delta \times \overset{\circ}{\mathbb{D}}_\rho$; it suffices to show that $SS_q(\mathbf{A}^\bullet) = \{0\}$.

In the following, we use the real structure in each of the statements. Proposition 8.5.8 of [6] implies Proposition 5.4.17 of [6]. Item ii, part d, of this latter proposition tells us that

$$SS_q(\mathbf{A}^\bullet) \subseteq \bigcup_{p \in B_\epsilon \cap g^{-1}(u_0) \cap f^{-1}(v_0)} \{ad_p \tilde{g} + bd_p \tilde{f} \in SS_p((\mathbf{F}^\bullet)_{\delta, \rho}^\epsilon) \mid (a, b) \in \mathbb{C}^2\}.$$

If $p \in \overset{\circ}{B}_\epsilon$, then $SS_p((\mathbf{F}^\bullet)_{\delta, \rho}^\epsilon) = SS_p(\mathbf{F}^\bullet)$, and so $\{ad_p \tilde{g} + bd_p \tilde{f} \in SS_p((\mathbf{F}^\bullet)_{\delta, \rho}^\epsilon) \mid (a, b) \in \mathbb{C}^2\} = \{0\}$ as $p \notin \overline{\Sigma_{\mathbf{F}^\bullet}(\tilde{f}, \tilde{g})}$. If $p \in \partial B_\epsilon$, then, by Proposition 5.4.8 of [6], $SS_p((\mathbf{F}^\bullet)_{\delta, \rho}^\epsilon) \subseteq SS_p(\mathbf{F}^\bullet) + \langle d_p r \rangle$, and so $\{ad_p \tilde{g} + bd_p \tilde{f} \in SS_p((\mathbf{F}^\bullet)_{\delta, \rho}^\epsilon) \mid (a, b) \in \mathbb{C}^2\} = \{0\}$ by Item 2 of Lemma 4.12. \square

Definition 4.14. *We refer to \mathbf{A}^\bullet in Theorem 4.13, for ϵ, δ , and ρ as in the theorem, as the **discriminant complex of (g, f) with respect to \mathbf{F}^\bullet** .*

Remark 4.15. The assumption that $\dim_{\mathbf{0}} V(f) \cap |\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)| \leq 0$ is crucial in Lemma 4.12 and Theorem 4.13. Consider the classic example of the map $H := (\tilde{g}, \tilde{f}) = (g, f) : \mathbb{C}^3 \rightarrow \mathbb{C}^2$ given by $g(x, y, t) = x$ and $f(x, y, t) = y^2 - tx^2$, where it is not possible to stratify the domain and codomain in order to obtain a Thom map. The (ordinary) discriminant of H is simply the origin and, yet, for $0 < \delta, \rho \ll \epsilon \ll 1$, the isomorphism-type of the cohomology of the fibers $B_\epsilon \cap H^{-1}(a, b)$ is not independent of the choice of $(a, b) \in \mathring{\mathbb{D}}_\delta \times \mathring{\mathbb{D}}_\rho - \{\mathbf{0}\}$.

The reader should verify that, in this example, $|\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)| = V(x, y)$ and so the condition that $\dim_{\mathbf{0}} V(f) \cap |\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)| \leq 0$ does not hold.

The following corollary is the technical justification for being able to prove results “upstairs” by using arguments “downstairs”.

Corollary 4.16. *Let $\mathcal{W} := \mathring{\mathbb{D}}_\delta \times \mathring{\mathbb{D}}_\rho$ (from Theorem 4.13). Let $D := \mathring{\mathbb{D}}_\delta \times \{0\}$, and denote other irreducible components of $(\mathring{\mathbb{D}}_\delta \times \mathring{\mathbb{D}}_\rho) \cap \Delta_{\mathbf{F}^\bullet}(\tilde{g}, \tilde{f})$ by C . Let $T_D^* \mathcal{W} := \overline{T_{D-\{0\}}^* \mathcal{W}}$ and $T_C^* \mathcal{W} := \overline{T_{C-\{0\}}^* \mathcal{W}}$. Let \mathbf{A}^\bullet be as in Theorem 4.13.*

Then, using (u, v) as coordinates on \mathcal{W} , for all k , we have isomorphisms

1.
$$H^k(\phi_u[-1]\psi_v[-1]\mathbf{A}^\bullet)_{\mathbf{0}} \cong H^k(\phi_g[-1]\psi_f[-1]\mathbf{F}^\bullet)_{\mathbf{0}},$$

2.
$$H^k(\phi_v[-1]\psi_u[-1]\mathbf{A}^\bullet)_{\mathbf{0}} \cong H^k(\phi_f[-1]\psi_g[-1]\mathbf{F}^\bullet)_{\mathbf{0}},$$

3.
$$H^k(\psi_u[-1]\phi_v[-1]\mathbf{A}^\bullet)_{\mathbf{0}} \cong H^k(\psi_g[-1]\phi_f[-1]\mathbf{F}^\bullet)_{\mathbf{0}},$$

4.
$$\mathbb{H}^{k-1}(F_{v,\mathbf{0}}, F_{v|_{V(u)}, \mathbf{0}}; \mathbf{A}^\bullet)_{\mathbf{0}} \cong \mathbb{H}^{k-1}(F_{f,\mathbf{0}}, F_{f|_{V(g)}, \mathbf{0}}; \mathbf{F}^\bullet),$$

and

5.
$$\mathbb{H}^{k-1}(F_{u,\mathbf{0}}, F_{u|_{V(v)}, \mathbf{0}}; \mathbf{A}^\bullet)_{\mathbf{0}} \cong \mathbb{H}^{k-1}(F_{g,\mathbf{0}}, F_{g|_{V(f)}, \mathbf{0}}; \mathbf{F}^\bullet).$$

In addition,

$$\text{gecc}^k(\mathbf{A}^\bullet) = m_{\mathcal{W}}^k[T_{\mathcal{W}}^* \mathcal{W}] + m_{\{\mathbf{0}\}}^k[T_{\{\mathbf{0}\}}^* \mathcal{W}] + m_D^k[T_D^* \mathcal{W}] + \sum_C m_C^k[T_C^* \mathcal{W}],$$

where $m_D^k \cong H^k(\psi_g[-1]\phi_f[-1]\mathbf{F}^\bullet)_{\mathbf{0}}$, and the m_C^k are determined by the fact that $\sum_C m_C^k[C]$ is equal to the proper push-forward, via (g, f) , of the enriched cycle $(\Gamma_{f,\tilde{g}}^1(\mathbf{F}^\bullet))^k$ (intersected with $N_{\delta,\rho}^\epsilon$).

Proof. Recall the set-up from Theorem 4.13. For simplicity, we will write N in place of $N_{\delta,\rho}^\epsilon$, T in place of $T_{\delta,\rho}^\epsilon$, and \mathbf{F}^\bullet in place of $(\mathbf{F}^\bullet)_{\delta,\rho}^\epsilon$. We assume that $\epsilon > 0$ is chosen so small that, below, for all of the complexes on subsets of X , the stalk cohomology at the origin is isomorphic to the hypercohomology inside B_ϵ .

The first five isomorphisms all follow from standard natural isomorphisms, combined with the fact that T is proper. We will prove isomorphism 5, and leave the remainder as exercises.

Let i be the inclusion of $N - V(f)$ into N , \hat{i} the inclusion of $\mathcal{W} - V(v)$ into \mathcal{W} , and \hat{T} the restriction of T to a map from $N - V(f)$ to $\mathcal{W} - V(v)$. Hence, the maps i , T , \hat{i} , and \hat{T} form a Cartesian square. Both T

and \hat{T} are proper, and so there are natural isomorphisms $T|_! \cong T_*$ and $\hat{T}|_! \cong \hat{T}_*$. Let \check{T} denote the restriction of T to a map from $N \cap V(g)$ to $V(u)$.

Now, we have

$$\begin{aligned} \mathbb{H}^{k-1}(F_{u,\mathbf{0}}, F_{u|_{V(v)}}, \mathbf{0}; \mathbf{A}^\bullet)_{\mathbf{0}} &\cong H^k(\psi_u[-1]i_!i^!\mathbf{A}^\bullet)_{\mathbf{0}} \cong H^k(\psi_u[-1]i_!\hat{T}_*\mathbf{F}^\bullet)_{\mathbf{0}} \cong H^k(\psi_u[-1]i_!\hat{T}_*i^!\mathbf{F}^\bullet)_{\mathbf{0}} \cong \\ &H^k(\psi_u[-1]i_!\hat{T}|_!i^!\mathbf{F}^\bullet)_{\mathbf{0}} \cong H^k(\psi_u[-1](i \circ \hat{T})_!i^!\mathbf{F}^\bullet)_{\mathbf{0}} \cong H^k(\psi_u[-1](T \circ i)_!i^!\mathbf{F}^\bullet)_{\mathbf{0}} \cong H^k(\psi_u[-1]T_*i_!i^!\mathbf{F}^\bullet)_{\mathbf{0}} \cong \\ &H^k(\check{T}_*\psi_{u \circ T}[-1]i_!i^!\mathbf{F}^\bullet)_{\mathbf{0}} = H^k(\check{T}_*\psi_g[-1]i_!i^!\mathbf{F}^\bullet)_{\mathbf{0}} \cong \mathbb{H}^k(B_\epsilon \cap V(g, f); \psi_g[-1]i_!i^!\mathbf{F}^\bullet) \cong H^k(\psi_g[-1]i_!i^!\mathbf{F}^\bullet)_{\mathbf{0}} \cong \\ &\mathbb{H}^{k-1}(F_{g,\mathbf{0}}, F_{g|_{V(f)}}, \mathbf{0}; \mathbf{F}^\bullet). \end{aligned}$$

This proves the isomorphism in Item 5.

We will now prove the claim about the graded, enriched characteristic cycle of \mathbf{A}^\bullet . By definition, $m_D^k \cong H^k(\psi_u[-1]\phi_v[-1]\mathbf{A}^\bullet)_{\mathbf{0}}$, which is isomorphic to $H^k(\psi_g[-1]\phi_f[-1]\mathbf{F}^\bullet)_{\mathbf{0}}$ by Item 3.

Let C be as in the statement of the theorem, and consider a point $(a, b) \in C - \{\mathbf{0}\}$. As the intersection of C and $V(v - b)$ is transverse at (a, b) , m_C^k is isomorphic to

$$H^k(\phi_{u-a}[-1](\mathbf{A}^\bullet_{|_{V(v-b)}}[-1]))_{(a,b)} \cong H^k(\phi_{u-a}[-1]\psi_{v-b}[-1]\mathbf{A}^\bullet)_{(a,b)}.$$

By Item 1, this last quantity is isomorphic to

$$\bigoplus H^k(\phi_{g-a}[-1]\psi_{f-b}[-1]\mathbf{F}^\bullet)_p,$$

where the sum is over all $p \in N \cap \text{supp } \phi_{g-a}[-1]\psi_{f-b}[-1]\mathbf{F}^\bullet$. By Theorem 3.12, this is isomorphic to

$$\bigoplus ((\Gamma_{f,\bar{g}}^1(\mathbf{F}^\bullet))^k \odot V(f - b))_p,$$

where $p \in N \cap |\Gamma_{f,\bar{g}}^1(\mathbf{F}^\bullet)| \cap V(g - a, f - b)$. This is precisely the definition of the degree k Morse module of C in the proper push-forward, via (g, f) , of the enriched cycle $(\Gamma_{f,\bar{g}}^1(\mathbf{F}^\bullet))^k$. \square

Definition 4.17. *The graded, enriched cycle $\sum_C m_C^\bullet[C]$ defined above is the **graded, enriched Cerf diagram of (g, f) with respect to \mathbf{F}^\bullet .***

*The graded, enriched cycle $m_D^\bullet[D] + \sum_C m_C^\bullet[C]$ defined above is the **graded, enriched discriminant of (g, f) with respect to \mathbf{F}^\bullet .***

Given the previous lemma, the proof of the following lemma proceeds exactly like classical discriminant/Cerf diagram arguments.

Corollary 4.18. *Suppose that $\dim_{\mathbf{0}} V(f) \cap |\Gamma_{f,\bar{g}}^1(\mathbf{F}^\bullet)| \leq 0$. Let $(\widehat{\Gamma}_{f,\bar{g}}^1(\mathbf{F}^\bullet))^\bullet$ denote the components of $(\Gamma_{f,\bar{g}}^1(\mathbf{F}^\bullet))^\bullet$ which are not contained in $V(g)$. Then,*

1.

$$\mathbb{H}^{k-1}(F_{f,\mathbf{0}}, F_{f|_{V(g)}}, \mathbf{0}; \mathbf{F}^\bullet) \cong ((\widehat{\Gamma}_{f,\bar{g}}^1(\mathbf{F}^\bullet))^k \odot V(f))_{\mathbf{0}};$$

2.

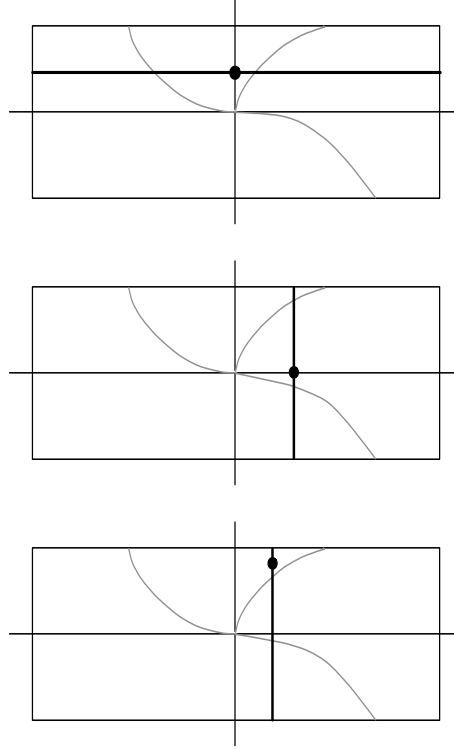
$$\mathbb{H}^{k-1}(F_{g,\mathbf{0}}, F_{g|_{V(f)}}, \mathbf{0}; \mathbf{F}^\bullet) \cong ((\widehat{\Gamma}_{f,\bar{g}}^1(\mathbf{F}^\bullet))^k \odot V(g))_{\mathbf{0}}; \text{ and}$$

3.

$$H^k(\phi_f[-1]\psi_g[-1]\mathbf{F}^\bullet)_0 \cong ((\widehat{\Gamma}_{f,\bar{g}}^1(\mathbf{F}^\bullet))^k \odot V(g))_0 \oplus H^k(\psi_g[-1]\phi_f[-1]\mathbf{F}^\bullet)_0.$$

Proof. Now that we have Theorem 4.13, the proof of each item is obtained by looking at the relative hypercohomology of a complex disk modulo a point, and using that this relative hypercohomology splits as a direct sum. One “sees” the results by looking at “pictures” in $\mathring{\mathbb{D}}_\delta \times \mathring{\mathbb{D}}_\rho$; exactly as in the case where \mathbf{F}^\bullet is the constant sheaf on affine space and g is a generic linear form. The discriminant/Cerf diagram arguments remain the same, **except** that it is no longer true that the components of the Cerf diagram are tangent to the horizontal axis at the origin, i.e., it is not necessarily true for each component C of $|\widehat{\Gamma}_{f,\bar{g}}(\mathbf{F}^\bullet)^\bullet|$ that $(C \cdot V(f))_0 > (C \cdot V(g))_0$.

Of course, the pictures are actually drawn in \mathbb{R}^2 , and so a line segment represents a complex disk (but a point still represents a point). The three relevant pictures, in order, are:



□

Remark 4.19. If $\dim_0 V(g) \cap |\Gamma_{f,\bar{g}}(\mathbf{F}^\bullet)| \leq 0$, then $(\widehat{\Gamma}_{f,\bar{g}}(\mathbf{F}^\bullet))^\bullet = (\Gamma_{f,\bar{g}}(\mathbf{F}^\bullet))^\bullet$, and Item 1 of Corollary 4.18, combined with Theorem 3.12, yields an isomorphism between the cohomology $\mathbb{H}^*(F_{f,0}, F_{f|_{V(g)},0}; \mathbf{F}^\bullet[-1])$ and $H^*(\phi_g[-1]\psi_f[-1]\mathbf{F}^\bullet)_0$; this isomorphism is **not** natural. In particular, the Milnor monodromy of f typically induces completely different automorphisms on these two cohomologies.

Corollary 4.20. *The following are equivalent:*

1. $0 \notin |\Gamma_{f,\bar{g}}(\mathbf{F}^\bullet)|$;
2. for all $p \in V(f, g)$ near the origin, $H^*(\phi_g[-1]\psi_f[-1]\mathbf{F}^\bullet)_p = 0$;

3. $\dim_{\mathbf{0}} V(f) \cap |\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)| \leq 0$, and $H^*(\phi_g[-1]\psi_f[-1]\mathbf{F}^\bullet)_{\mathbf{0}} = 0$;
4. $\dim_{\mathbf{0}} V(g) \cap |\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)| \leq 0$, and inclusion induces an isomorphism between $\mathbb{H}^*(F_{f,\mathbf{0}}; \mathbf{F}^\bullet)$ and $\mathbb{H}^*(F_{f|_{V(g)}}, \mathbf{0}; \mathbf{F}^\bullet)$;
5. $\dim_{\mathbf{0}} V(g) \cap |\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)| \leq 0$, and inclusion induces an isomorphism between $\mathbb{H}^*(F_{g,\mathbf{0}}; \mathbf{F}^\bullet)$ and $\mathbb{H}^*(F_{g|_{V(f)}}, \mathbf{0}; \mathbf{F}^\bullet)$;
6. $\dim_{\mathbf{0}} V(g) \cap |\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)| \leq 0$, and $H^*(\phi_f[-1]\psi_g[-1]\mathbf{F}^\bullet)_{\mathbf{0}}$ is isomorphic to $H^*(\psi_g[-1]\phi_f[-1]\mathbf{F}^\bullet)_{\mathbf{0}}$.

Proof. This is immediate from Lemma 3.10, Theorem 3.12, and Corollary 4.18. \square

The above corollary has a more familiar feel in the classical case, where \tilde{g} is a generic linear form. Recall that, in Proposition 3.13, we showed that, if \mathfrak{l} is a generic linear form on \mathcal{U} , then, $\dim_{\mathbf{0}} V(f) \cap |\Gamma_{f,\mathfrak{l}}(\mathbf{F}^\bullet)| \leq 0$ and $\dim_{\mathbf{0}} V(\mathfrak{l}) \cap |\Gamma_{f,\mathfrak{l}}(\mathbf{F}^\bullet)| \leq 0$. Thus, Corollary 4.20 immediately yields:

Corollary 4.21. *The following are equivalent:*

1. there exists a non-zero linear form \mathfrak{l} such that $\mathbf{0} \notin |\Gamma_{f,\mathfrak{l}}(\mathbf{F}^\bullet)|$;
2. for generic linear \mathfrak{l} , $\mathbf{0} \notin |\Gamma_{f,\mathfrak{l}}(\mathbf{F}^\bullet)|$;
3. for generic linear \mathfrak{l} , $H^*(\phi_{\mathfrak{l}}[-1]\psi_f[-1]\mathbf{F}^\bullet)_{\mathbf{0}} = 0$;
4. for generic linear \mathfrak{l} , inclusion induces an isomorphism between $\mathbb{H}^*(F_{f,\mathbf{0}}; \mathbf{F}^\bullet)$ and $\mathbb{H}^*(F_{f|_{V(\mathfrak{l})}}, \mathbf{0}; \mathbf{F}^\bullet)$;
5. for generic linear \mathfrak{l} , inclusion induces an isomorphism between $\mathbb{H}^*(F_{\mathfrak{l},\mathbf{0}}; \mathbf{F}^\bullet)$ and $\mathbb{H}^*(F_{\mathfrak{l}|_{V(f)}}, \mathbf{0}; \mathbf{F}^\bullet)$;
6. for generic linear \mathfrak{l} , $H^*(\phi_f[-1]\psi_{\mathfrak{l}}[-1]\mathbf{F}^\bullet)_{\mathbf{0}}$ is isomorphic to $H^*(\psi_{\mathfrak{l}}[-1]\phi_f[-1]\mathbf{F}^\bullet)_{\mathbf{0}}$.

Remark 4.22. Suppose that $X = \mathcal{U}$ and $\mathbf{F}^\bullet = \mathbb{Z}_{\mathcal{U}}^\bullet$ throughout this remark.

Then, $\mathbb{H}^*(F_{\mathfrak{l},\mathbf{0}}; \mathbf{F}^\bullet)$ has the cohomology of a point and $\mathbb{H}^*(F_{f|_{V(\mathfrak{l})}}, \mathbf{0}; \mathbf{F}^\bullet)$ is the cohomology of the complex link of $V(f)$ at $\mathbf{0}$. Hence, the equivalence of Item 2 (or Item 1) and Item 5 of Corollary 4.21 is a generalization of the well-known result that the cohomology of the complex link of $V(f)$ at $\mathbf{0}$ is isomorphic to that of a point if and only if the relative polar curve is empty (in a neighborhood of $\mathbf{0}$).

Suppose now that we also have that $\dim_{\mathbf{0}} \Sigma f = 1$. Then, for generic \mathfrak{l} , $f|_{V(\mathfrak{l})}$ has an isolated critical point at the origin, and $H^*(\phi_f[-1]\psi_{\mathfrak{l}}[-1]\mathbf{F}^\bullet)_{\mathbf{0}}$ is isomorphic to the (shifted) reduced integral cohomology of the Milnor fiber at the origin of $f|_{V(\mathfrak{l})}$. On the other hand, $H^*(\psi_{\mathfrak{l}}[-1]\phi_f[-1]\mathbf{F}^\bullet)_{\mathbf{0}}$ is isomorphic to the direct sum of the reduced integral cohomologies of the Milnor fibers of $f|_{V(\mathfrak{l}-t)}$, where the sum is over all points $p \in \overset{\circ}{B}_\epsilon \cap \Sigma f \cap V(\mathfrak{l}-t)$ for $0 < |t| \ll \epsilon \ll 1$. Therefore, the equivalence of Item 2 (or Item 1) and Item 6 of Corollary 4.21 is a generalization of the well-known result that the Milnor number of a generic hyperplane slice equals the sum of the Milnor numbers in a nearby hyperplane slice if and only if the relative polar curve is empty (in a neighborhood of $\mathbf{0}$).

5 An Application to Thom's a_f Condition

By combining our results from [19] with Theorem 3.12, we can relate the polar curve to Thom's a_f condition. Essentially what we prove below, in Theorem 5.7, is that, if a stratification satisfies the a_f condition, except perhaps at a point p on a 1-dimensional stratum, then the stratification satisfies the a_f condition at p if and only if, for some affine linear form \mathfrak{l} , the polar curve of (f, \mathfrak{l}) at p is empty.

However, we do not actually need to start with a stratification, for we do not need the condition of the frontier. Also, of course, we want such a result with respect to a complex of sheaves. So, we need to make a number of preliminary definitions before we can state and prove our precise result.

Suppose that M and N are complex submanifolds of \mathcal{U} .

Definition 5.1. *The pair (M, N) satisfies Thom's $a_{\tilde{f}}$ condition at a point $x \in N$ if and only if there is an inclusion, of fibers over x , $(\overline{T_{\tilde{f}|_M}^* \mathcal{U}})_x \subseteq (T_{\tilde{f}|_N}^* \mathcal{U})_x$.*

The pair (M, N) satisfies Thom's $a_{\tilde{f}}$ condition if and only if it satisfies the $a_{\tilde{f}}$ condition at each point $x \in N$.

Remark 5.2. Note that if \tilde{f} is a locally constant function, then the $a_{\tilde{f}}$ condition reduces to condition (a) of Whitney.

The $a_{\tilde{f}}$ condition is important for several reasons. First, it is an hypothesis of Thom's second isotopy lemma; see [20]. Second, the $a_{\tilde{f}}$ condition, and the existence of stratifications in which all pairs of strata satisfy the $a_{\tilde{f}}$ condition, is essential in arguments such as that used by Lê in [9] to prove that Milnor fibrations exist even when the domain is an arbitrarily singular space. Third, the $a_{\tilde{f}}$ condition is closely related to constancy of the Milnor number in families of isolated hypersurface singularities; see [11].

There are at least two important general results about the $a_{\tilde{f}}$ condition: the above-mentioned existence of $a_{\tilde{f}}$ stratifications, proved first by Hironaka in [5] and then in a different manner by Hamm and Lê, following an argument of F. Pham, in Theorem 1.2.1 of [4], and the fact that Whitney stratifications in which $V(\tilde{f}) := \tilde{f}^{-1}(0)$ is a union of strata are $a_{\tilde{f}}$ stratifications, proved independently by Parusiński in [21], and Briançon, P. Maisonobe, and M. Merle in [1].

We can easily prove the following:

Proposition 5.3. *Suppose that $p \notin \overline{\Sigma_{\mathbf{F}^\bullet} f}$. Let M and N be analytic submanifolds of \mathcal{U} such that $\overline{T_M^* \mathcal{U}}$ is an irreducible component of $SS(\mathbf{F}^\bullet)$, and such that (M, N) satisfies Whitney's condition (a) at a point $p \in N$.*

Then, $(\overline{T_{\tilde{f}|_M}^ \mathcal{U}})_p \subseteq (T_N^* \mathcal{U})_p + \langle d_p \tilde{f} \rangle$. In particular, if \tilde{f} is locally constant on N at p or if $p \notin \Sigma(\tilde{f}|_N)$, then (M, N) satisfies Thom's $a_{\tilde{f}}$ condition at p .*

Proof. By replacing \mathcal{U} with a small neighborhood of p , we may assume that N is connected and closed in \mathcal{U} . Note that, as $\overline{T_M^* \mathcal{U}}$ is an irreducible component of $SS(\mathbf{F}^\bullet)$, M is connected, $M \subseteq X$, and $\tilde{f}|_M = f|_M$.

There are 2 cases to consider.

Case 1: \tilde{f} is constant on M .

By Whitney's condition (a),

$$(\overline{T_{\tilde{f}|_M}^* \mathcal{U}})_p = (\overline{T_M^* \mathcal{U}})_p \subseteq (T_N^* \mathcal{U})_p \subseteq (T_N^* \mathcal{U})_p + \langle d_p \tilde{f} \rangle.$$

Case 2: \tilde{f} is not constant on M .

Since $p \notin \overline{\Sigma_{\mathbf{F}^\bullet} f}$ and $\overline{T_M^* \mathcal{U}}$ is an irreducible component of $SS(\mathbf{F}^\bullet)$, Lemma 4.8 tells us that $(\overline{T_{\tilde{f}|_M}^* \mathcal{U}})_p = (\overline{T_M^* \mathcal{U}})_p + \langle d_p \tilde{f} \rangle$, and so, using Whitney's condition (a) again, we have

$$(\overline{T_{\tilde{f}|_M}^* \mathcal{U}})_p = (\overline{T_M^* \mathcal{U}})_p + \langle d_p \tilde{f} \rangle \subseteq (T_N^* \mathcal{U})_p + \langle d_p \tilde{f} \rangle.$$

□

Definition 5.4. A collection \mathcal{W} of subsets of X is a **(complex analytic) partition** of X if and only if \mathcal{W} is a locally finite disjoint collection of analytic submanifolds of \mathcal{U} , which we call strata, whose union is all of X , and such that, for each stratum $W \in \mathcal{W}$, \overline{W} and $\overline{W} - W$ are closed complex analytic subsets of X .

In this paper, we assume that all of the strata of a partition are connected.

A partition \mathcal{W} is a **stratification** if and only if it satisfies the condition of the frontier, i.e., for all $W \in \mathcal{W}$, \overline{W} is a union of elements of \mathcal{W} .

Below, we extend our earlier definition of \mathbf{F}^\bullet -visible strata to the case of a partition which may not satisfy Whitney conditions.

Definition 5.5. A partition \mathcal{W} of X is an **\mathbf{F}^\bullet -partition** provided that

$$SS(\mathbf{F}^\bullet) \subseteq \bigcup_{W \in \mathcal{W}} \overline{T_W^* \mathcal{U}}.$$

If \mathcal{W} is an **\mathbf{F}^\bullet -partition**, then a stratum $W \in \mathcal{W}$ is **\mathbf{F}^\bullet -visible** if and only if $\overline{T_W^* \mathcal{U}} \subseteq SS(\mathbf{F}^\bullet)$. We let $\mathcal{W}(\mathbf{F}^\bullet) := \{W \in \mathcal{W} \mid W \text{ is } \mathbf{F}^\bullet\text{-visible}\}$.

Remark 5.6. The reader should understand that the point of an **\mathbf{F}^\bullet -partition** \mathcal{W} is that, for each **\mathbf{F}^\bullet -visible** stratum S in \mathfrak{S} , there exists a unique $W \in \mathcal{W}$ such that $\overline{S} = \overline{W}$ and, hence, $\overline{T_S^* \mathcal{U}} = \overline{T_W^* \mathcal{U}}$. It follows at once from this, and the definition of **\mathbf{F}^\bullet -visible** strata of \mathcal{W} , that, if \mathcal{W} is an **\mathbf{F}^\bullet -partition**, then

$$SS(\mathbf{F}^\bullet) = \bigcup_{W \in \mathcal{W}(\mathbf{F}^\bullet)} \overline{T_W^* \mathcal{U}}.$$

We can now give our result on Thom's a_f condition and the relative polar curve.

Theorem 5.7. Suppose that

- \mathcal{W} is an **\mathbf{F}^\bullet -partition** of X ;
- \mathcal{W}' is a Whitney (a) partition of $V(f)$;
- $\mathbf{0} \in T \in \mathcal{W}'$ and $\dim T = 1$;
- for all $W \in \mathcal{W}(\mathbf{F}^\bullet)$ such that $W \not\subseteq V(f)$, for all $W' \in \mathcal{W}'$ such that $W' \neq T$, (W, W') satisfies the a_f condition.

Then, the condition:

(\dagger) for all $W \in \mathcal{W}(\mathbf{F}^\bullet)$ such that $W \not\subseteq V(f)$, (W, T) satisfies the a_f condition

is equivalent to all of the conditions in Corollary 4.21; in particular, it is equivalent to: there exists a non-zero linear form \mathfrak{l} such that $\mathbf{0} \notin |\Gamma_{f, \mathfrak{l}}(\mathbf{F}^\bullet)|$.

Proof. Let $\mathcal{W}'_T := \{W' \in \mathcal{W}' \mid W' \neq T\}$. By Corollary 3.9 of [19], our hypotheses imply that, if $p \in W' \in \mathcal{W}'_T$, and \mathfrak{l}' is a non-zero linear form such that $V(\mathfrak{l}' - \mathfrak{l}'(p))$ transversely intersects W' at p , then $p \notin \text{supp } \phi_{\mathfrak{l}' - \mathfrak{l}'(p)}[-1]\psi_f[-1]\mathbf{F}^\bullet$.

Assume (\dagger). By Corollary 3.9 of [19], $\psi_f[-1]\mathbf{F}^\bullet$ is ϕ -constructible with respect to \mathcal{W}' . In particular, this implies that, if \mathfrak{l} is a non-zero linear form such that $V(\mathfrak{l})$ transversely intersects T at $\mathbf{0}$, then $\mathbf{0} \notin \text{supp } \phi_{\mathfrak{l}}[-1]\psi_f[-1]\mathbf{F}^\bullet$. This implies Item 3 of Corollary 4.21.

Assume Item 3 of Corollary 4.21. Fix a non-zero \mathfrak{l} such that $V(\mathfrak{l})$ transversely intersects T at $\mathbf{0}$ and $H^*(\phi_{\mathfrak{l}}[-1]\psi_f[-1]\mathbf{F}^\bullet)_{\mathbf{0}} = 0$. As \mathcal{W}' satisfies Whitney (a), $V(\mathfrak{l})$ transversely intersects all of the strata of \mathcal{W}' in a neighborhood of $\mathbf{0}$. Thus, by the first paragraph of the proof, $\mathbf{0} \notin \text{supp } \phi_{\mathfrak{l}}[-1]\psi_f[-1]\mathbf{F}^\bullet$. As T is 1-dimensional, this, together with the first paragraph of the proof, implies that $\psi_f[-1]\mathbf{F}^\bullet$ is weakly ϕ -constructible with respect to \mathcal{W}' . By Corollary 3.9 of [19], this implies (\dagger). \square

6 Families of Isolated Critical Points

In [14] and [17], we discussed continuous families of constructible complexes of sheaves. We wish to revisit our results in those works, and show how the proofs and results can be greatly improved by using the main theorems of this paper.

We continue to let \tilde{f} and \tilde{g} be analytic functions from \mathcal{U} to \mathbb{C} , and we let f and g be their respective restrictions to the analytic space X . We continue with \mathbf{F}^\bullet being a bounded, constructible complex of sheaves of R -modules on X .

Throughout this section, we consider that \mathbf{F}^\bullet and g define a family (a g -family) of constructible complexes by setting $\mathbf{F}_a^\bullet := \mathbf{F}^\bullet|_{V(g-a)}[-1]$, for each $a \in \mathbb{C}$. We also consider the family of functions $f_a := f|_{V(g-a)}$. As in [14] and [17], we make the following definition.

Definition 6.1. *The family \mathbf{F}_a^\bullet is continuous at $a = a_0$ if and only if $\phi_{g-a_0}[-1]\mathbf{F}^\bullet = 0$. The family \mathbf{F}_a^\bullet is continuous at $p \in X$ if and only if there exists an open neighborhood of p in which $\phi_{g-g(p)}[-1]\mathbf{F}^\bullet = 0$.*

The following is an additivity/upper-semicontinuity result. Recall our definition of the critical locus of a function relative to a complex of sheaves from Definition 4.2.

Proposition 6.2. *Suppose that the family \mathbf{F}_a^\bullet is continuous at $\mathbf{0}$, and that $\dim_0 \overline{\Sigma_{\mathbf{F}_0^\bullet} f_0} \leq 0$. Then, $\dim_0 \overline{\Sigma_{\mathbf{F}^\bullet} f} \leq 1$, and there exists an open neighborhood Ω of $\mathbf{0}$ in \mathcal{U} and $\delta > 0$ such that, if $|a| < \delta$, then $\Omega \cap \overline{\Sigma_{\mathbf{F}_a^\bullet} f_a}$ is either empty or consists of a finite number of points and, for all k ,*

$$H^k(\phi_{f_0}[-1]\mathbf{F}_0^\bullet)_{\mathbf{0}} \cong \bigoplus_{p \in \Omega \cap \overline{\Sigma_{\mathbf{F}_a^\bullet} f_a}} H^k(\phi_{f_a}[-1]\mathbf{F}_a^\bullet)_p.$$

Proof. As $\phi_g[-1]\mathbf{F}^\bullet = 0$ in a neighborhood of $\mathbf{0}$, $\phi_{f_0}[-1]\mathbf{F}_0^\bullet = \phi_f[-1]\psi_g[-1]\mathbf{F}^\bullet$ near $\mathbf{0}$, and so the assumption that $\dim_0 \overline{\Sigma_{\mathbf{F}_0^\bullet} f_0} \leq 0$ is equivalent to $\dim_{\mathbf{0}} \text{supp } \phi_f[-1]\psi_g[-1]\mathbf{F}^\bullet \leq 0$. By Theorem 3.12, this implies that $\dim_{\mathbf{0}} V(g) \cap |\Gamma_{g,\tilde{f}}(\mathbf{F}^\bullet)| \leq 0$ and

$$H^k(\phi_{f_0}[-1]\mathbf{F}_0^\bullet)_{\mathbf{0}} \cong H^k(\phi_f[-1]\psi_g[-1]\mathbf{F}^\bullet)_{\mathbf{0}} \cong ((\Gamma_{g,\tilde{f}}^1(\mathbf{F}^\bullet))^k \odot V(g))_{\mathbf{0}}.$$

Hence, there exists an open neighborhood Ω of $\mathbf{0}$ in \mathcal{U} and $\delta > 0$ such that, if $|a| < \delta$, then $\Omega \cap V(g-a) \cap |\Gamma_{g,\tilde{f}}(\mathbf{F}^\bullet)|$ is either empty or consists of a finite number of points, and

$$H^k(\phi_{f_0}[-1]\mathbf{F}_0^\bullet)_{\mathbf{0}} \cong \bigoplus_{p \in \Omega \cap V(g-a) \cap |\Gamma_{g,\tilde{f}}(\mathbf{F}^\bullet)|} ((\Gamma_{g,\tilde{f}}^1(\mathbf{F}^\bullet))^k \odot V(g-a))_p \cong \bigoplus_{p \in \Omega \cap \overline{\Sigma_{\mathbf{F}_a^\bullet} f_a}} H^k(\phi_{f_a}[-1]\mathbf{F}_a^\bullet)_p,$$

where the last equality follows again from Theorem 3.12, and using that, near $\mathbf{0}$, $\overline{\Sigma_{\mathbf{F}^\bullet} g} \subseteq V(g)$. \square

There is the trivial case:

Proposition 6.3. *Suppose that the family \mathbf{F}_a^\bullet is continuous at $\mathbf{0}$, and that $\mathbf{0} \notin \overline{\Sigma_{\mathbf{F}_0^\bullet} f_0}$. Then,*

1. $\mathbf{0} \notin \overline{\Sigma_{\mathbf{F}^\bullet} f}$;
2. near $\mathbf{0}$, $\phi_g[-1](\psi_f[-1]\mathbf{F}^\bullet) = \phi_g[-1](\mathbf{F}^\bullet|_{V(f)}[-1]) = 0$, i.e., $\psi_f[-1]\mathbf{F}^\bullet$, and $\mathbf{F}^\bullet|_{V(f)}[-1]$ are g -continuous families at $\mathbf{0}$; and

3. if M and N are analytic submanifolds of \mathcal{U} such that $\overline{T_M^* \mathcal{U}}$ is an irreducible component of $SS(\mathbf{F}^\bullet)$, such that (M, N) satisfies Whitney's condition (a) at $\mathbf{0}$, and such that $N \subseteq V(f)$, or $\mathbf{0} \notin \Sigma(f|_N)$, then (M, N) satisfies Thom's $a_{\tilde{f}}$ condition at $\mathbf{0}$.

Proof.

Proof of Item 1:

By Proposition 4.10,

$$\overline{\Sigma_{\mathbf{F}^\bullet} g} \cup |\Gamma_{g, \tilde{f}}(\mathbf{F}^\bullet)| = \overline{\Sigma_{\mathbf{F}^\bullet}(\tilde{f}, \tilde{g})} = \overline{\Sigma_{\mathbf{F}^\bullet} f} \cup |\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)|.$$

As $\mathbf{0} \notin \overline{\Sigma_{\mathbf{F}^\bullet} g}$, there is an equality of sets $|\Gamma_{g, \tilde{f}}(\mathbf{F}^\bullet)| = \overline{\Sigma_{\mathbf{F}^\bullet} f} \cup |\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)|$ near $\mathbf{0}$. As $\mathbf{0} \notin \overline{\Sigma_{\mathbf{F}_0^\bullet} f_0}$, Proposition 6.2 implies that $\mathbf{0} \notin |\Gamma_{g, \tilde{f}}(\mathbf{F}^\bullet)|$. Item 1 follows.

Proof of Item 2:

It follows from the paragraph above that $\mathbf{0} \notin |\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)|$. As $|\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)|$ is closed, we may apply Theorem 3.12 at each point p near the origin in $V(f, g)$ to conclude that, near $\mathbf{0}$, $\phi_g[-1](\psi_f[-1]\mathbf{F}^\bullet) = 0$. By Item 1, $\phi_f[-1]\mathbf{F}^\bullet = 0$ near the origin, and so $\psi_f[-1]\mathbf{F}^\bullet = \mathbf{F}_{|_{V(f)}}^\bullet[-1]$ near $\mathbf{0}$. Item 2 follows.

Proof of Item 3:

This follows immediately from Item 1 and Proposition 5.3. \square

The following theorem contains generalizations of well-known properties/results that hold in the classic case of families of isolated critical points of functions on affine space, including a generalization of the main result of Lê and Saito from [11].

Theorem 6.4. *Suppose that the family \mathbf{F}_a^\bullet is continuous at $\mathbf{0}$, and that $\dim_0 \overline{\Sigma_{\mathbf{F}_0^\bullet} f_0} \leq 0$.*

Let $C \subseteq V(f)$ be a locally irreducible curve which contains $\mathbf{0}$ such that, for all $p \in C$ near $\mathbf{0}$, the isomorphism-type of $H^(\phi_{f_{g(p)}}[-1]\mathbf{F}_{g(p)}^\bullet)_p$ is non-zero and equal to $H^*(\phi_{f_0}[-1]\mathbf{F}_0^\bullet)_\mathbf{0}$. Then, in a neighborhood of the origin,*

1. C is smooth;
2. $V(\tilde{g})$ is smooth, and transversely intersects C ;
3. $\mathbf{0} \notin |\Gamma_{f, \tilde{g}}(\mathbf{F}^\bullet)|$;
4. $C = \overline{\Sigma_{\mathbf{F}^\bullet} f} = |\Gamma_{g, \tilde{f}}(\mathbf{F}^\bullet)|$;
5. $\phi_g[-1](\phi_f[-1]\mathbf{F}^\bullet) = 0$, $\phi_g[-1](\psi_f[-1]\mathbf{F}^\bullet) = 0$, and $\phi_g[-1](\mathbf{F}_{|_{V(f)}}^\bullet[-1]) = 0$, i.e., $\phi_f[-1]\mathbf{F}^\bullet$, $\psi_f[-1]\mathbf{F}^\bullet$, and $\mathbf{F}_{|_{V(f)}}^\bullet[-1]$ are g -continuous families at $\mathbf{0}$;
6. $SS(\phi_f[-1]\mathbf{F}^\bullet) = T_C^* \mathcal{U}$;
7. for all k , the sheaf cohomology $\mathbf{H}^{k-1}(\phi_f[-1]\mathbf{F}^\bullet)$ is constant on C and, for all $p \in C$, there is an isomorphism of stalk cohomology $H^{k-1}(\phi_f[-1]\mathbf{F}^\bullet)_p \cong H^k(\phi_{f_0}[-1]\mathbf{F}_0^\bullet)_\mathbf{0}$;
8. for all $p \in C$, the Milnor monodromy automorphisms

$$\tilde{T}_{f_{g(p)}, p}^* : H^*(\phi_{f_{g(p)}}[-1]\mathbf{F}_{g(p)}^\bullet)_p \rightarrow H^*(\phi_{f_{g(p)}}[-1]\mathbf{F}_{g(p)}^\bullet)_p$$

are isomorphic to $\tilde{T}_{f_0, \mathbf{0}}^*$, i.e., there are isomorphisms between $H^*(\phi_{f_{g(p)}}[-1]\mathbf{F}_{g(p)}^\bullet)_p$ and $H^*(\phi_{f_0}[-1]\mathbf{F}_0^\bullet)_\mathbf{0}$ which commute with the monodromy automorphisms; and

9. if M is an analytic submanifold of \mathcal{U} such that $\overline{T_M^* \mathcal{U}}$ is an irreducible component of $SS(\mathbf{F}^\bullet)$ and (M, C) satisfies Whitney's condition (a) at $\mathbf{0}$, then (M, C) satisfies Thom's a_f condition at $\mathbf{0}$.

Proof. As we saw in Proposition 6.2, the assumptions imply that there exists an open neighborhood Ω of $\mathbf{0}$ in \mathcal{U} and $\delta > 0$ such that, if $|a| < \delta$,

$$H^k(\phi_{f_0}[-1]\mathbf{F}_0^\bullet)_{\mathbf{0}} \cong \bigoplus_{p \in \Omega \cap V(g-a) \cap |\Gamma_{g,\tilde{f}}(\mathbf{F}^\bullet)|} ((\Gamma_{g,\tilde{f}}^1(\mathbf{F}^\bullet))^k \odot V(g-a))_p \cong \bigoplus_{p \in \Omega \cap \overline{\Sigma_{\mathbf{F}^\bullet} f_a}} H^k(\phi_{f_a}[-1]\mathbf{F}_a^\bullet)_p.$$

Hence, the constancy of the isomorphism-type of $H^*(\phi_{f_{g(p)}}[-1]\mathbf{F}_{g(p)}^\bullet)$ along C implies that, near $\mathbf{0}$, $C = |\Gamma_{g,\tilde{f}}(\mathbf{F}^\bullet)|$, C is smooth at $\mathbf{0}$, and $V(\tilde{g})$ is smooth at $\mathbf{0}$, and transversely intersects C at $\mathbf{0}$. This proves Items 1 and 2.

As we saw in the proof of Proposition 6.3, near $\mathbf{0}$, $|\Gamma_{g,\tilde{f}}(\mathbf{F}^\bullet)| = \overline{\Sigma_{\mathbf{F}^\bullet} f} \cup |\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)|$; as we saw above, this equals the irreducible curve C . As $\dim_{\mathbf{0}} V(g) \cap |\Gamma_{g,\tilde{f}}(\mathbf{F}^\bullet)| = 0$, $\dim_{\mathbf{0}} V(g) \cap |\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)| \leq 0$. By Lemma 3.10, this implies that $\dim_{\mathbf{0}} V(f) \cap |\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)| \leq 0$. As every component of $|\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)|$ has dimension at least one, and as $|\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)|$ is contained in $C \subseteq V(f)$ near $\mathbf{0}$, we conclude that $\mathbf{0} \notin |\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)|$; this is Item 3. As $|\Gamma_{g,\tilde{f}}(\mathbf{F}^\bullet)| = \overline{\Sigma_{\mathbf{F}^\bullet} f} \cup |\Gamma_{f,\tilde{g}}(\mathbf{F}^\bullet)| = C$ near $\mathbf{0}$, we conclude Item 4.

From Item 3 of this theorem and Theorem 3.12, we conclude that the stalk cohomology at the origin of $\phi_g[-1](\psi_f[-1]\mathbf{F}^\bullet)$ is equal to zero. As we may apply this argument at each point near $\mathbf{0}$, we conclude that $\phi_g[-1](\psi_f[-1]\mathbf{F}^\bullet) = 0$ near $\mathbf{0}$. From Item 3 of this theorem and Item 2 of Corollary 4.18, we find that $\mathbb{H}^*(F_{g,\mathbf{0}}, F_{g|_{V(f)}, \mathbf{0}}; \mathbf{F}^\bullet) = 0$. As $\mathbb{H}^*(B_\epsilon, F_{g,\mathbf{0}}; \mathbf{F}^\bullet) = 0$, we conclude that $\mathbb{H}^*(B_\epsilon \cap V(f), F_{g|_{V(f)}, \mathbf{0}}; \mathbf{F}^\bullet) = 0$, i.e., that the stalk cohomology at $\mathbf{0}$ of $\phi_g[-1](\mathbf{F}_{|_{V(f)}}^\bullet[-1])$ is zero. Since we may apply this argument at each point near $\mathbf{0}$, we conclude that $\phi_g[-1](\mathbf{F}_{|_{V(f)}}^\bullet[-1]) = 0$ near the origin. As $\phi_g[-1](\psi_f[-1]\mathbf{F}^\bullet) = 0$ and $\phi_g[-1](\mathbf{F}_{|_{V(f)}}^\bullet[-1]) = 0$ near $\mathbf{0}$, the distinguished triangle relating the nearby and vanishing cycles implies that $\phi_g[-1](\phi_f[-1]\mathbf{F}^\bullet) = 0$ near $\mathbf{0}$. This proves Item 5.

Item 6 follows at once from Items 4 and 5, together with the fact that $d_{\mathbf{0}}\tilde{g} \neq 0$ by Item 2. Item 7 follows immediately from Item 4 and the fact, from Item 5, that $\phi_g[-1](\phi_f[-1]\mathbf{F}^\bullet) = 0$ near $\mathbf{0}$.

Items 3 and 5 tell us that, in Theorem 4.13, $\mathbf{A}^\bullet := R(T_{\delta,\rho}^\epsilon)_*(\mathbf{F}^\bullet)_{\delta,\rho}^\epsilon$ is complex analytically constructible with respect to the stratification given by $\{\mathring{\mathbb{D}}_\delta \times \mathring{\mathbb{D}}_\rho - \mathring{\mathbb{D}}_\delta \times \{0\}, \mathring{\mathbb{D}}_\delta \times \{0\}\}$. Using (u, v) for coordinates on $\mathring{\mathbb{D}}_\delta \times \mathring{\mathbb{D}}_\rho$, it is trivial that the Milnor monodromy on $H^*(\phi_v[-1](\mathbf{A}_{|_{V(u-a)}}^\bullet)_{(a,0)})$ is constant (up to isomorphism) for small $|a|$. As $C = \overline{\Sigma_{\mathbf{F}^\bullet} f} = \text{supp } \phi_f[-1]\mathbf{F}^\bullet$ near $\mathbf{0}$, Item 8 follows.

Let M be as in Item 9, and let $Y := \overline{M}$. If $\mathbf{0} \notin Y$ or if $Y = C$ near $\mathbf{0}$, then Item 9 follows trivially. So, suppose that $\mathbf{0} \in Y$ and $Y \neq C$ near $\mathbf{0}$. As $\overline{T_M^* \mathcal{U}}$ is an irreducible component of $SS(\mathbf{F}^\bullet)$, Y is irreducible. If f were constant on Y , then, as $\mathbf{0} \in Y$, Y would have to be contained in $V(f)$, and so we would have $Y \subseteq \overline{\Sigma_{\mathbf{F}^\bullet} f}$; this is impossible by Item 4.

Now, let E denote the exceptional divisor in the blow-up $\text{Bl}_{\text{im } d\tilde{f}}(\overline{T_M^* \mathcal{U}}) \subseteq T^*\mathcal{U} \times \mathbb{P}^n$. We identify $T^*\mathcal{U} \times \mathbb{P}^n$ with $\mathcal{U} \times \mathbb{C}^{n+1} \times \mathbb{P}^n$, and let $\sigma : \mathcal{U} \times \mathbb{C}^{n+1} \times \mathbb{P}^n \rightarrow \mathcal{U} \times \mathbb{P}^n$ denote the projection. Over a neighborhood of the origin, Item 6 tells us that $SS(\phi_f[-1]\mathbf{F}^\bullet) = T_C^*\mathcal{U}$. Now, by Theorem 3.4 of [18], $\sigma(E) \cap (\{\mathbf{0}\} \times \mathbb{P}^n) \subseteq \mathbb{P}(T_C^*\mathcal{U})_{\mathbf{0}}$. By Proposition 4.3 of [16], this implies that (M, C) satisfies Thom's a_f condition at $\mathbf{0}$. \square

Remark 6.5. The reader should compare the statements and the proofs from Theorem 6.4 with our related results in [14] and [17]. Not only are our current results more general and stronger, the proofs are vastly easier.

We also remark that Item 5 of Theorem 6.4 is useful for inductions. For instance, in proving results for families of local complete intersections with isolated singularities.

Recall the definition of the graded, enriched characteristic cycle from Definition 2.4. Let $SS^k(\mathbf{F}^\bullet)$ equal the underlying set $|\text{gecc}^k(\mathbf{F}^\bullet)|$. This means that $SS^k(\mathbf{F}^\bullet) = \bigcup \overline{T_S^* \mathcal{U}}$, where the union is over those strata $S \in \mathfrak{S}$ such that $H^{k-d_S}(\mathbb{N}_S, \mathbb{L}_S; \mathbf{F}^\bullet) \neq 0$. We also set $\Sigma_{\mathbf{F}^\bullet}^k f := \{p \in X \mid H^k(\phi_{f-f(p)}[-1]\mathbf{F}^\bullet)_p \neq 0\}$.

In Proposition 2.5 of [18], we explained how the perverse cohomology operator ${}^\mu H^k$ works with graded, enriched characteristic cycles; we showed there is an equality of enriched cycles given by

$$\text{gecc}^0({}^\mu H^k(\mathbf{F}^\bullet)) = \text{gecc}^k(\mathbf{F}^\bullet).$$

This is a consequence of the fact that the perverse cohomology operator commutes with the $[-1]$ -shifted vanishing cycle operator, and that, if p is an isolated point in the support of a complex \mathbf{A}^\bullet , then $H^j({}^\mu H^k(\mathbf{A}^\bullet))_p$ is isomorphic to $H^k(\mathbf{A}^\bullet)_p$ when $j = 0$, and is zero if $j \neq 0$.

The following corollary, which tells us that Theorem 6.4 applies one degree at a time, follows immediately by replacing \mathbf{F}^\bullet by ${}^\mu H^k(\mathbf{F}^\bullet)$ in each line, other than the first, in Theorem 6.4. Note, however, that the conditions in the first line are **not** conditions on a fixed degree.

Corollary 6.6. *Suppose that the family \mathbf{F}_a^\bullet is continuous at $\mathbf{0}$, and that $\dim_0 \overline{\Sigma_{\mathbf{F}_0^\bullet} f_0} \leq 0$. Fix $k \in \mathbb{Z}$.*

Let $C \subseteq V(f)$ be a locally irreducible curve which contains $\mathbf{0}$ such that, for all $p \in C$ near $\mathbf{0}$, the isomorphism-type of $H^k(\phi_{f_{g(p)}}[-1]\mathbf{F}_{g(p)}^\bullet)_p$ is non-zero and equal to $H^k(\phi_{f_0}[-1]\mathbf{F}_0^\bullet)_\mathbf{0}$. Then, in a neighborhood of the origin,

1. C is smooth;
2. $V(\tilde{g})$ is smooth, and transversely intersects C ;
3. $\mathbf{0} \notin |(\Gamma_{f,\tilde{g}}^1(\mathbf{F}^\bullet))^k|$;
4. $C = \overline{\Sigma_{\mathbf{F}^\bullet}^{k-1} f} = |(\Gamma_{g,\tilde{f}}^1(\mathbf{F}^\bullet))^k|$;
5. $SS^k(\phi_f[-1]\mathbf{F}^\bullet) = T_C^* \mathcal{U}$;
6. the sheaf cohomology $\mathbf{H}^{k-1}(\phi_f[-1]\mathbf{F}^\bullet)$ is constant on C and, for all $p \in C$, there is an isomorphism of stalk cohomology $H^{k-1}(\phi_f[-1]\mathbf{F}^\bullet)_p \cong H^k(\phi_{f_0}[-1]\mathbf{F}_0^\bullet)_\mathbf{0}$;
7. for all $p \in C$, the Milnor monodromy automorphisms

$$\tilde{T}_{f_{g(p)},p}^k : H^k(\phi_{f_{g(p)}}[-1]\mathbf{F}_{g(p)}^\bullet)_p \rightarrow H^k(\phi_{f_{g(p)}}[-1]\mathbf{F}_{g(p)}^\bullet)_p$$

are isomorphic to $\tilde{T}_{f_0,\mathbf{0}}^k$, i.e., there are isomorphisms between $H^k(\phi_{f_{g(p)}}[-1]\mathbf{F}_{g(p)}^\bullet)_p$ and $H^k(\phi_{f_0}[-1]\mathbf{F}_0^\bullet)_\mathbf{0}$ which commute with the monodromy automorphisms; and

8. *if M is an analytic submanifold of \mathcal{U} such that $\overline{T_M^* \mathcal{U}}$ is an irreducible component of $SS^k(\mathbf{F}^\bullet)$ and (M, C) satisfies Whitney's condition (a) at $\mathbf{0}$, then (M, C) satisfies Thom's a_f condition at $\mathbf{0}$.*

References

- [1] Briançon, J., Maisonobe, P., and Merle, M. Localisation de systèmes différentiels, stratifications de Whitney et condition de Thom. *Invent. Math.*, 117:531–550, 1994.
- [2] Fulton, W. *Intersection Theory*, volume 2 of *Ergeb. Math.* Springer-Verlag, 1984.
- [3] Goresky, M. and MacPherson, R. *Stratified Morse Theory*, volume 14 of *Ergeb. der Math.* Springer-Verlag, 1988.

- [4] Hamm, H. and Lê D. T. Un théorème de Zariski du type de Lefschetz. *Ann. Sci. Éc. Norm. Sup.*, 6 (series 4):317–366, 1973.
- [5] Hironaka, H. Stratification and flatness. In P. Holm, editor, *Real and Complex Singularities, Oslo 1976*, pages 199–265. Nordic Summer School/NAVF, 1977.
- [6] Kashiwara, M. and Schapira, P. *Sheaves on Manifolds*, volume 292 of *Grund. math. Wissen.* Springer-Verlag, 1990.
- [7] Lê, D. T. Calcul du Nombre de Cycles Evanouissants d’une Hypersurface Complexe. *Ann. Inst. Fourier, Grenoble*, 23:261–270, 1973.
- [8] Lê, D. T. Topological Use of Polar Curves. *Proc. Symp. Pure Math.*, 29:507–512, 1975.
- [9] Lê, D. T. Some remarks on Relative Monodromy. In P. Holm, editor, *Real and Complex Singularities, Oslo 1976*, pages 397–404. Nordic Summer School/NAVF, 1977.
- [10] Lê, D. T. Sur les cycles évanouissants des espaces analytiques. *C. R. Acad. Sci. Paris, Sér. A-B*, 288:A283–A285, 1979.
- [11] Lê, D. T. and Saito, K. La constance du nombre de Milnor donne des bonnes stratifications. *C.R. Acad. Sci.*, 277:793–795, 1973.
- [12] Massey, D. *Lê Cycles and Hypersurface Singularities*, volume 1615 of *Lecture Notes in Math.* Springer-Verlag, 1995.
- [13] Massey, D. Hypercohomology of Milnor Fibres. *Topology*, 35:969–1003, 1996.
- [14] Massey, D. Critical Points of Functions on Singular Spaces. *Top. and Appl.*, 103:55–93, 2000.
- [15] Massey, D. A Little Microlocal Morse Theory. *Math. Ann.*, 321:275–294, 2001.
- [16] Massey, D. Perverse Cohomology and the Vanishing Index Theorem. *Top. and Appl.*, 125:299–313, 2002.
- [17] Massey, D. *Numerical Control over Complex Analytic Singularities*, volume 778 of *Memoirs of the AMS.* AMS, 2003.
- [18] Massey, D. Singularities and Enriched Cycles. *Pacific J. Math.*, 215, no. 1:35–84, 2004.
- [19] Massey, D. Vanishing Cycles and Thom’s a_f Condition. [math.AG/0605369](https://arxiv.org/abs/math/0605369), 2006.
- [20] Mather, J. Notes on Topological Stability. Notes from Harvard Univ., 1970.
- [21] Parusiński, A. Limits of Tangent Spaces to Fibres and the w_f Condition. *Duke Math. J.*, 72:99–108, 1993.
- [22] Sabbah, C. Proximité évanescence. *Compos. Math.*, 62:283–328, 1987.
- [23] Teissier, B. Cycles évanescents, sections planes et conditions de Whitney. *Astérisque*, 7-8:285–362, 1973.