

Hypersurface Singularities and the Swing*

Lê Dũng Tráng and David B. Massey

Abstract

Suppose that f defines a singular, complex affine hypersurface. If the critical locus of f is one-dimensional, we obtain new general bounds on the ranks of the homology groups of the Milnor fiber of f . This result has an interesting implication on the structure of the vanishing cycles in the category of perverse sheaves.

1 Introduction and Previous Results

Let \mathcal{U} be an open neighborhood of the origin in \mathbb{C}^{n+1} , and let $f : (\mathcal{U}, \mathbf{0}) \rightarrow (\mathbb{C}, 0)$ be complex analytic. We shall always suppose that $\dim_{\mathbf{0}} \Sigma f = 1$, unless we explicitly state otherwise.

Let $F_f = F_{f, \mathbf{0}}$ denote the Milnor fiber of f at the origin. It is well-known (see [6]) that the reduced integral homology, $\tilde{H}_*(F_f)$, of F_f can be non-zero only in degrees $n - 1$ and n , and is free Abelian in degree n . For arbitrary f , it is not known how to calculate, algebraically, the groups $\tilde{H}_{n-1}(F_f)$ and $\tilde{H}_n(F_f)$; in fact, it is not known how to calculate the ranks of these groups. However, there are a number of general results known for these “top” two homology groups of F_f .

First, we need to make some choices and establish some notation.

We assume that the first coordinate z_0 on \mathcal{U} is a generic linear form; in the terminology of [11], we need for z_0 to be “prepolar” (with respect to f at the origin). This implies that, at the origin, $f_0 := f|_{V(z_0)}$ has an isolated critical point, that the polar curve, $\Gamma := \Gamma_{f, z_0}^1$, is purely 1-dimensional at the origin (which vacuously includes the case $\Gamma = \emptyset$), and Γ has no components contained in $V(f)$ (this last property is immediate in some definitions of the relative polar curve).

For convenience, we assume throughout the remainder of this paper that the neighborhood \mathcal{U} is re-chosen, if necessary, so small that $\Sigma f \subseteq V(f)$, and every component of Σf and Γ contains the origin.

Now, there is the attaching result of Lê from [9] (see, also, [11]), which is valid regardless of the dimension of the critical locus:

Theorem 1.1. *Up to diffeomorphism, F_f is obtained from $\mathbb{D} \times_{F_{f_0}}^{\circ}$ by attaching $\tau := (\Gamma \cdot V(f))_{\mathbf{0}}$ handles of index n .*

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Remark 1.2. On the level of homology, Lê's attaching result is a type of Lefschetz hyperplane result; it says that, for all $i < n - 1$, the inclusion map $F_{f_0} = F_f \cap V(z_0) \hookrightarrow F_f$ induces isomorphisms $\tilde{H}_i(F_{f_0}) \cong \tilde{H}_i(F_f)$, and $\tilde{H}_n(F_f)$ and $\tilde{H}_{n-1}(F_f)$ are, respectively, isomorphic to the kernel and cokernel of the boundary map

$$\mathbb{Z}^\tau \cong H_n(F_f, F_{f_0}) \xrightarrow{\partial} \tilde{H}_{n-1}(F_{f_0}) \cong \mathbb{Z}^{\mu_{f_0}},$$

where μ_{f_0} denotes the Milnor number of f_0 at the origin. Therefore, one can certainly calculate the difference of the reduced Betti numbers of F_f :

$$\tilde{b}_n(F_f) - \tilde{b}_{n-1}(F_f) = \tau - \mu_{f_0}.$$

Hence, bounds on one of $\tilde{b}_n(F_f)$ and $\tilde{b}_{n-1}(F_f)$ automatically produce bounds on the other.

We remind the reader here of the well-known result, first proved by Teissier in [15] (in the case of an isolated singularity, but the proof works in general), that

$$\tau = (\Gamma \cdot V(f))_{\mathbf{0}} = \left(\Gamma \cdot V \left(\frac{\partial f}{\partial z_0} \right) \right)_{\mathbf{0}} + (\Gamma \cdot V(z_0))_{\mathbf{0}}.$$

As defined in [11], the first summand on the right above is $\lambda^0 := \lambda_{f, z_0}^0(\mathbf{0})$, the 0-dimensional Lê number, and second summand on the right above is $\gamma^1 := \gamma_{f, z_0}^1(\mathbf{0})$, the 1-dimensional polar number.

For each component ν of Σf , let $\overset{\circ}{\mu}_\nu$ denote the Milnor number of $f|_{V(z_0-a)}$ at a point close to the origin on $\nu \cap V(z_0 - a)$, where a is a small non-zero complex number. Then,

$$\lambda^1 := \lambda_{f, z_0}^1(\mathbf{0}) := \sum_{\nu} \overset{\circ}{\mu}_\nu (\nu \cdot V(z_0))_{\mathbf{0}}$$

is the 1-dimensional Lê number of f . Now, it is well-known, and easy to show that $\mu_{f_0} = \gamma^1 + \lambda^1$. Again, see [11] for the above definitions and results.

In Proposition 3.1 of [11], the second author showed how the technique of “tilting in the Cerf diagram” or “the swing”, as used by Lê and Perron in [10] could help refine the result of Theorem 1.1. Here, we state only the homological implication of Proposition 3.1 of [11].

Theorem 1.3. *The boundary map $H_n(F_f, F_{f_0}) \xrightarrow{\partial} \tilde{H}_{n-1}(F_{f_0})$ maps a direct summand of $H_n(F_f, F_{f_0})$ of rank γ^1 isomorphically onto a direct summand of $\tilde{H}_{n-1}(F_{f_0})$.*

Thus, the rank of $\tilde{H}_n(F_f)$ is at most λ^0 , and the rank of $\tilde{H}_{n-1}(F_f)$ is at most λ^1 .

However, if one of the components ν of Σf is itself singular, then the above bounds on the ranks are known not to be optimal. A result of Siersma in [14], or an easy exercise using perverse sheaves (see the remark at the end of [14]), yields:

Theorem 1.4. *The rank of $\tilde{H}_{n-1}(F_f)$ is at most $\sum_{\nu} \overset{\circ}{\mu}_\nu$.*

Of course, if all of the components ν of Σf are smooth, and z_0 is generic, then $\lambda^1 = \sum_{\nu} \overset{\circ}{\mu}_{\nu}$, and the bounds on the ranks obtained from Theorem 1.3 and Theorem 1.4 are the same. In addition, Theorem 1.4 is true with arbitrary field coefficients; this yields bounds on the possible torsion in $\tilde{H}_{n-1}(F_f)$. We should also remark that the result of Siersma from [14] that we cite above can actually yield a much stronger bound if one knows certain extra topological data – specifically, one needs that the “vertical monodromies” are non-trivial.

Now, in light of Theorem 1.3 and Theorem 1.4, the question is: Is it possible that $\text{rank } \tilde{H}_{n-1}(F_f) = \lambda^1$?

Of course, the answer to this question is “yes”; if f has a smooth critical locus which defines a family of isolated singularities with constant Milnor number μ_{f_0} , then certainly $\tilde{H}_n(F_f) = 0$ and $\tilde{H}_{n-1}(F_f) \cong \mathbb{Z}^{\lambda^1} = \mathbb{Z}^{\mu_{f_0}}$. We refer to this case as the **trivial case**. It is important to note that being in the trivial case implies that $V(z_0)$ transversely intersects the smooth critical locus at the origin.

By the non-splitting result, proved independently by Gabrielov [5], Lazzeri [7], and Lê [8], we have:

Proposition 1.5. *The trivial case is equivalent to the case $\Gamma = \emptyset$.*

Now, we can state our Main Theorem:

Main Theorem. *Suppose that $\dim_{\mathbf{0}} \Sigma f = 1$ and $\dim_{\mathbf{0}} \Sigma f_0 = 0$. Then, the following are equivalent:*

- a) *We are in the trivial case, i.e., f has a smooth critical locus which defines a family of isolated singularities with constant Milnor number μ_{f_0} ;*
- b) $\text{rank } \tilde{H}_{n-1}(F_f) = \lambda^1$;
- c) *there exists a prime p such that $\dim \tilde{H}_{n-1}(F_f; \mathbb{Z}/p\mathbb{Z}) = \lambda^1$.*

Thus, if we are not in the trivial case, $\text{rank } \tilde{H}_{n-1}(F_f) < \lambda^1$, and so $\text{rank } \tilde{H}_n(F_f) < \lambda^0$, and these inequalities hold with $\mathbb{Z}/p\mathbb{Z}$ coefficients (here, p is prime).

Remark 1.6. We remark again that if one of the components of Σf is itself singular (and, hence, we are not in the trivial case), then the conclusion that $\text{rank } \tilde{H}_{n-1}(F_f) < \lambda^1$ already follows from Theorem 1.4. Even in the case where all of the components of Σf are smooth, we could conclude that $\text{rank } \tilde{H}_{n-1}(F_f) < \lambda^1$ from [14] **if** we knew that one of the vertical monodromies were non-trivial. However, the vertical monodromies are fairly complicated topological data to calculate, and it is also true that the vertical monodromies can be trivial even when the polar curve is non-empty, i.e., when we are not in the trivial case. Thus, our Main Theorem cannot be proved by analyzing the vertical monodromies.

In [13], Siersma proved another closely related result. On the level of homology, what he proved was that, if we are not in the trivial case, and Σf has a single smooth component, ν , such that $\overset{\circ}{\mu}_{\nu} = 1$, then $\tilde{H}_{n-1}(F_f) = 0$; our Main Theorem, including the modulo p statement, is a strict generalization of this.

In addition, we should point out that, in [3], Th. de Jong provides evidence that a result like our Main Theorem might be true.

We prove our Main Theorem by combining the swing technique of Theorem 1.3 and the connectivity of the vanishing cycle intersection diagram for isolated singularities, as was proved independently by Gabrielov in [5] and Lazzeri in [7]. In some recent notes, M. Tibăr uses similar techniques and reaches a number of conclusions closely related to our result.

As a corollary to our Main Theorem, we show that it implies that the vanishing cycles of f , as an object in the category of perverse sheaves, cannot be semi-simple in non-trivial cases where Σf has smooth components of arbitrary dimension.

In the final section of this paper, we make some final remarks and present counterexamples to some conceivable “improvements” on the statement of the Main Theorem.

2 The Swing

In the Introduction, we referred to the swing (or, tilting in the Cerf diagram), which was used by Lê and Perron in [10] and in Proposition 3.1 of [11], where the swing was used to prove Theorem 1.3. The swing has also been studied in [2], [16], [11], [17]. As the swing is so crucial to the proof of the main theorem, we wish to give a careful explanation of its construction.

Suppose that \mathcal{W} is an open neighborhood of the origin in \mathbb{C}^2 . We will use the coordinates x and y on \mathcal{W} . For notational ease, when we restrict x and y to various subspaces where the domain is clear, we shall continue to write simply x and y .

Let C be a complex analytic curve in \mathcal{W} such that every component of C contains the origin. We assume that the origin is an isolated point in $V(x) \cap C$ and in $V(y) \cap C$, i.e., that C does not have a component along the x - or y -axis.

Below, we let \mathbb{D}_ϵ denote a closed disk, of radius ϵ , centered at the origin, in the complex plane. We denote the interior of \mathbb{D}_ϵ by $\overset{\circ}{\mathbb{D}}_\epsilon$, and when we delete the origin, we shall superscript with an asterisk, i.e., $\mathbb{D}_\epsilon^* := \mathbb{D}_\epsilon - \{0\}$ and $\overset{\circ}{\mathbb{D}}_\epsilon^* := \overset{\circ}{\mathbb{D}}_\epsilon - \{0\}$.

We select $0 < \epsilon_2 \ll \epsilon_1 \ll 1$ so that:

- i): the “half-open” polydisk $\mathbb{D}_{\epsilon_1} \times \overset{\circ}{\mathbb{D}}_{\epsilon_2}$ is contained in \mathcal{W} ;
- ii): $(\partial\mathbb{D}_{\epsilon_1} \times \overset{\circ}{\mathbb{D}}_{\epsilon_2}) \cap C = \emptyset$ (this uses that the origin is an isolated point in $V(y) \cap C$);

Note that ii) implies that $(\mathbb{D}_{\epsilon_1} \times \overset{\circ}{\mathbb{D}}_{\epsilon_2}) \cap C = (\overset{\circ}{\mathbb{D}}_{\epsilon_1} \times \overset{\circ}{\mathbb{D}}_{\epsilon_2}) \cap C$.

- iii): $\mathbb{D}_{\epsilon_1} \times \overset{\circ}{\mathbb{D}}_{\epsilon_2}^* \xrightarrow{y} \overset{\circ}{\mathbb{D}}_{\epsilon_2}^*$ is a proper stratified submersion, where the Whitney strata are $\partial\mathbb{D}_{\epsilon_1} \times \overset{\circ}{\mathbb{D}}_{\epsilon_2}^*$, $(\overset{\circ}{\mathbb{D}}_{\epsilon_1} \times \overset{\circ}{\mathbb{D}}_{\epsilon_2}^*) - C$, and $(\overset{\circ}{\mathbb{D}}_{\epsilon_1} \times \overset{\circ}{\mathbb{D}}_{\epsilon_2}^*) \cap C$.

- iv): $(\overset{\circ}{\mathbb{D}}_{\epsilon_1} \times \overset{\circ}{\mathbb{D}}_{\epsilon_2}^*) \cap C \xrightarrow{y} \overset{\circ}{\mathbb{D}}_{\epsilon_2}^*$ is an m -fold covering map, where $m := (C \cdot V(y))_{\mathbf{0}}$.

Let $D := (\mathbb{D}_{\epsilon_1} \times \overset{\circ}{\mathbb{D}}_{\epsilon_2}) \cap (C \cup V(y))$. Let $(x_0, y_0) \in (\overset{\circ}{\mathbb{D}}_{\epsilon_1} \times \overset{\circ}{\mathbb{D}}_{\epsilon_2}^*) - D$. Let $\sigma : [0, 1] \rightarrow \{x_0\} \times \overset{\circ}{\mathbb{D}}_{\epsilon_2}$ be a smooth, simple path such that $\sigma(0) = (x_0, y_0)$, $\sigma(1) = (x_0, y_1) \in C$, and $\sigma([0, 1]) \subseteq (\{x_0\} \times \overset{\circ}{\mathbb{D}}_{\epsilon_2}) - D$.

Let S be the image of σ ; as σ is simple, S is homeomorphic to $[0, 1]$. Let $\sigma_0 := y \circ \sigma$ and let S_0 be the image of σ_0 . Thus, S_0 is homeomorphic to $[0, 1]$ and is contained in $\mathbb{D}_{\epsilon_2}^*$.

Lemma 2.1. (The Swing) *There exists a continuous function $H : [0, 1] \times [0, 1] \rightarrow \mathbb{D}_{\epsilon_1} \times S_0$ with the following properties:*

- a) $H(t, 0) = \sigma(t)$, for all $t \in [0, 1]$;
- b) $H(t, 1) \in \mathbb{D}_{\epsilon_1} \times \{y_0\}$, for all $t \in [0, 1]$;
- c) $H(0, u) = (x_0, y_0)$
- d) if $H(t, u) \in D$, then $t = 1$;
- e) $H(1, u) \in C$, for all $u \in [0, 1]$;
- f) the path η given by $\eta(u) := H(1, u)$ is a homeomorphism onto its image.

Thus, H is a homotopy from σ to the path γ given by $\gamma(t) := H(t, 1) \in \mathbb{D}_{\epsilon_1} \times \{y_0\}$, such that (x_0, y_0) is “fixed” and the point $(x_0, y_1) = H(1, 0)$ “swings up to the point” $H(1, 1)$ by “sliding along” C , while the remainder of σ does not hit D as it “swings up” to γ .

Proof. The proper stratified submersion $\mathbb{D}_{\epsilon_1} \times \mathbb{D}_{\epsilon_2}^* \xrightarrow{y} \mathbb{D}_{\epsilon_2}^*$ is a locally trivial fibration, where the local trivialization respects the strata. The restriction of this fibration $\mathbb{D}_{\epsilon_1} \times S_0 \xrightarrow{y} S_0$ is a locally trivial fibration over a contractible space and, hence, is equivalent to the trivial fibration.

Therefore, there exists a homeomorphism

$$\Psi : (\mathbb{D}_{\epsilon_1} \times S_0, (\mathbb{D}_{\epsilon_1} \times S_0) \cap C) \rightarrow (\mathbb{D}_{\epsilon_1} \times \{y_0\}, (\mathbb{D}_{\epsilon_1} \times \{y_0\}) \cap C) \times [0, 1]$$

such that the projection of $\Psi(x, \sigma_0(t))$ onto the $[0, 1]$ factor is simply t , and such that $\Psi(x, y_0) = ((x, y_0), 0)$. It follows that there is a path $\alpha : [0, 1] \rightarrow \mathbb{D}_{\epsilon_1}$ such that $\Psi(\sigma(t)) = ((\alpha(t), y_0), t)$, for all $t \in [0, 1]$. Define $H : [0, 1] \times [0, 1] \rightarrow \mathbb{D}_{\epsilon_1} \times S_0$ by

$$H(t, u) := \Psi^{-1}((\alpha(t), y_0), (1 - u)t).$$

All of the given properties of H are now trivial to verify. \square

Remark 2.2. By Property c) of Lemma 2.1, the map H yields a corresponding map H^T whose domain is a triangle instead of a square. One pictures the image of H , or of H^T , as a “gluing in” of this triangle into $\mathbb{D}_{\epsilon_1} \times S_0$ in such a way that one edge of the triangle is glued diffeomorphically to S , and another edge is glued diffeomorphically onto the image of η . The third edge of the triangle is glued onto the image of γ , but not necessarily in a one-to-one fashion.

3 The Main Theorem

In this section, we will prove the Main Theorem, as stated in the Introduction and as appears below as Theorem 3.1. That a) of the Main Theorem implies both b) and c) is well-known; one can, for instance, conclude

it from Theorem 1.1. The difficulty is to prove that b) and c) imply a). In fact, we prove the contrapositives; we prove that if we are not in the trivial case, then $\text{rank } \tilde{H}_{n-1}(F_f) < \lambda^1$ and $\dim \tilde{H}_{n-1}(F_f; \mathbb{Z}/p\mathbb{Z}) < \lambda^1$. We must first describe the machinery that goes into this part of the proof.

As the value of λ^1 is minimal for generic z_0 , we lose no generality if we assume that our linear form z_0 is chosen more generically than simply being prepolar. We choose z_0 so generically that, in addition to being prepolar, the discriminant, D , of the map (z_0, f) and the corresponding Cerf diagram, C , have the usual properties – as given, for instance, in [10], [16], and [17]. We will describe the needed properties below.

Let $\tilde{\Psi} := (z_0, f) : (\mathcal{U}, \mathbf{0}) \rightarrow (\mathbb{C}^2, \mathbf{0})$. We use the coordinates (u, v) on \mathbb{C}^2 . The critical locus $\Sigma\tilde{\Psi}$ of $\tilde{\Psi}$ is the union of Σf and Γ . The discriminant $D := \tilde{\Psi}(\Sigma\tilde{\Psi})$ consists of the u -axis together with the Cerf diagram $C := \overline{D - V(v)}$. We assume that z_0 is generic enough so that the polar curve is reduced and that, in a neighborhood of the origin, $\tilde{\Psi}|_{\Gamma}$ is one-to-one.

We choose real numbers ϵ, δ , and ω so that $0 < \omega \ll \delta \ll \epsilon \ll 1$. Let $B_\epsilon \subseteq \mathbb{C}^n$ be a closed ball, centered at the origin, of radius ϵ . Let $\mathring{\mathbb{D}}_\delta$ and $\mathring{\mathbb{D}}_\omega$ be open disks in \mathbb{C} , centered at 0, of radii δ and ω , respectively.

One considers the map from $(\mathring{\mathbb{D}}_\delta \times B_\epsilon) \cap f^{-1}(\mathring{\mathbb{D}}_\omega)$ onto $\mathring{\mathbb{D}}_\delta \times \mathring{\mathbb{D}}_\omega$ given by the restriction of $\tilde{\Psi}$; we let Ψ denote this restriction. As B_ϵ is a closed ball, the map Ψ is certainly proper, but the domain has an interior stratum, and a stratum coming from the boundary of B_ϵ . However, for generic z_0 , all of the stratified critical points lie on $\Gamma \cup \Sigma f$, i.e., above D .

We continue to write simply D and C , in place of $D \cap (\mathring{\mathbb{D}}_\delta \times \mathring{\mathbb{D}}_\omega)$ and $C \cap (\mathring{\mathbb{D}}_\delta \times \mathring{\mathbb{D}}_\omega)$. As Ψ is a proper stratified submersion above $\mathring{\mathbb{D}}_\delta \times \mathring{\mathbb{D}}_\omega - D$, and as $\Psi|_{\Gamma}$ is one-to-one, many homotopy arguments in $(\mathring{\mathbb{D}}_\delta \times B_\epsilon) \cap f^{-1}(\mathring{\mathbb{D}}_\omega)$ can be obtained from lifting constructions in $\mathring{\mathbb{D}}_\delta \times \mathring{\mathbb{D}}_\omega$. This is the point of considering the discriminant and Cerf diagram.

Let $v_0 \in \mathring{\mathbb{D}}_\omega - \{0\}$. By construction, up to diffeomorphism, $\Psi^{-1}(\mathring{\mathbb{D}}_\delta \times \{v_0\})$ is F_f and $\Psi^{-1}((0, v_0))$ is F_{f_0} . In fact, for all u_0 , where $|u_0| \ll |v_0|$, $\Psi^{-1}((u_0, v_0))$ is diffeomorphic to F_{f_0} ; we fix such a non-zero u_0 , and let $\mathbf{a} := (u_0, v_0)$.

We wish to pick a distinguished basis for the vanishing cycles of f_0 at the origin, as in I.1 of [1] (see, also, [4]). We do this by selecting paths in $\{u_0\} \times \mathring{\mathbb{D}}_\omega$ which originate at \mathbf{a} . We must be slightly careful in how we do this.

First, fix a path p_0 from \mathbf{a} to $(u_0, 0)$. Select paths q_1, \dots, q_{γ^1} from \mathbf{a} to each of the points in $(\{u_0\} \times \mathring{\mathbb{D}}_\omega) \cap C =: \{y_1, \dots, y_{\gamma^1}\}$. The paths $p_0, q_1, \dots, q_{\gamma^1}$ should not intersect each other and should intersect the set $\{(u_0, 0), y_1, \dots, y_{\gamma^1}\}$ only at the endpoints of the paths. Moreover, when at the point \mathbf{a} , the paths $p_0, q_1, \dots, q_{\gamma^1}$ should be in clockwise order. Let r_0 be a clockwise loop very close to p_0 , from \mathbf{a} around $(u_0, 0)$.

As we are not assuming that f had an isolated line singularity, we must perturb $f|_{V(z_0 - u_0)}$ slightly to have $(u_0, 0)$ split into λ^1 points, $x_1, \dots, x_{\lambda^1}$ inside the loop r_0 ; each of these points corresponds to an A_1 singularity in the domain. We select paths $p_1, \dots, p_{\lambda^1}$ from \mathbf{a} to each of the points $x_1, \dots, x_{\lambda^1}$, and paths q_1, \dots, q_{γ^1} from \mathbf{a} to each of the points in $(\{u_0\} \times \mathring{\mathbb{D}}_\omega) \cap C =: \{y_1, \dots, y_{\gamma^1}\}$. We may do this in such a way that the paths $p_1, \dots, p_{\lambda^1}, q_1, \dots, q_{\gamma^1}$ are in clockwise order.

The lifts of these paths via the perturbed $f|_{V(z_0 - u_0)}$ yield representatives of elements of $H_{n+1}(B_\epsilon, F_{f_0})$, whose boundaries in $\tilde{H}_n(F_{f_0})$ form a distinguished basis $\Delta'_1, \dots, \Delta'_{\lambda^1}, \Delta_1, \dots, \Delta_{\gamma^1}$.

By using the swing (Lemma 2.1), the paths q_1, \dots, q_{γ^1} are taken to new paths $\hat{q}_1, \dots, \hat{q}_{\gamma^1}$ in $\mathring{\mathbb{D}}_\delta \times \{v_0\}$. Each \hat{q}_i path represents a relative homology class in $H_n(F_f, F_{f_0})$ whose boundary in $\tilde{H}_{n-1}(F_{f_0})$ is precisely Δ_i . Theorem 1.3 follows from this.

We can now prove the Main Theorem:

Theorem 3.1. *Suppose that $\dim_0 \Sigma f = 1$ and $\dim_0 \Sigma f_0 = 0$. Then, the following are equivalent:*

a) *We are in the trivial case, i.e., f has a smooth critical locus which defines a family of isolated singularities with constant Milnor number μ_{f_0} ;*

b) $\text{rank } \tilde{H}_{n-1}(F_f) = \lambda^1$;

c) *there exists a prime p such that $\dim \tilde{H}_{n-1}(F_f; \mathbb{Z}/p\mathbb{Z}) = \lambda^1$.*

Thus, if we are not in the trivial case, $\text{rank } \tilde{H}_{n-1}(F_f) < \lambda^1$, and so $\text{rank } \tilde{H}_n(F_f) < \lambda^0$, and these inequalities hold with $\mathbb{Z}/p\mathbb{Z}$ coefficients (here, p is prime).

Proof. As mentioned above, that a) implies b) and c) is well-known. Assume then that we are not in the trivial case. We will prove that $\text{rank } \tilde{H}_{n-1}(F_f) < \lambda^1$, and then indicate why the same proof applies with $\mathbb{Z}/p\mathbb{Z}$ coefficients.

By Proposition 1.5, $\Gamma \neq \emptyset$, and so $C \neq \emptyset$. We want to construct just one new path in $\{u_0\} \times \mathring{\mathbb{D}}_\omega$, one which originates at \mathbf{a} , ends at a point of C , and misses all of the other points of D ; we want this path to swing up to a path in $\mathring{\mathbb{D}}_\delta \times \{v_0\}$, and represent a relative homology class in $H_n(F_f, F_{f_0})$ whose boundary is not in the span of $\Delta_1, \dots, \Delta_{\gamma^1}$.

By the connectivity of the vanishing cycle intersection diagram ([5], [7]), one of the Δ'_j must have a non-zero intersection pairing with one of the Δ_i , i.e., there exist i_0 and j_0 such that $\langle \Delta_{i_0}, \Delta'_{j_0} \rangle \neq 0$.

By fixing the path p_{j_0} and all the q_i paths, but reselecting the other p_j , for $j \neq j_0$, we may assume that $j_0 = 1$, i.e., that $\langle \Delta_{i_0}, \Delta'_1 \rangle \neq 0$.

We follow now Chapter 3.3 of [4]. Associated to each path p_j , $1 \leq j \leq \lambda^1$, is a (partial) monodromy automorphism $T'_j : \tilde{H}_{n-1}(F_{f_0}) \rightarrow \tilde{H}_{n-1}(F_{f_0})$, induced by taking a clockwise loop r_j very close to p_j , from \mathbf{a} around x_j . Let $T' := T'_1 \dots T'_{\lambda^1}$, where composition is written in the order of [4]. We claim that $T'(\Delta_{i_0})$ is in the image of $\delta : H_n(F_f, F_{f_0}) \rightarrow H_{n-1}(F_{f_0})$, but is not in $\text{Span}\{\Delta_1, \dots, \Delta_{\gamma^1}\}$.

The composition r of the loops $r_1, \dots, r_{\lambda^1}$ is homotopy-equivalent, in $\{u_0\} \times \mathring{\mathbb{D}}_\omega - \{\{x_1, \dots, x_{\lambda^1}\} \cup C\}$, to the loop r_0 (from our discussion before the theorem). By combining (concatenating) the loop r_0 and the path q_{i_0} , we obtain a path in $\{u_0\} \times \mathring{\mathbb{D}}_\omega$ which is homotopy-equivalent to a simple path which swings up to a corresponding path in $\mathring{\mathbb{D}}_\delta \times \{v_0\}$. Thus, $T'(\Delta_{i_0})$ is in the image of δ .

Now, by the Corollaries to the Picard-Lefschetz Theorem in [1], p. 26, or as in [4], Formula 3.11,

$$T'(\Delta_{i_0}) = \Delta_{i_0} - (-1)^{\frac{n(n-1)}{2}} \langle \Delta_{i_0}, \Delta'_1 \rangle \Delta'_1 + \beta_2 \Delta'_2 + \dots + \beta_{\lambda^1} \Delta'_{\lambda^1},$$

for some integers $\beta_2, \dots, \beta_{\lambda^1}$. As the $\Delta'_1, \dots, \Delta'_{\lambda^1}, \Delta_1, \dots, \Delta_{\gamma^1}$ form a basis, and as $\langle \Delta_{i_0}, \Delta'_1 \rangle \neq 0$, $T'(\Delta_{i_0})$ is not in $\text{Span}\{\Delta_1, \dots, \Delta_{\gamma^1}\}$.

This finishes the proof over the integers. Over $\mathbb{Z}/p\mathbb{Z}$, the proof is identical, since the intersection diagram is also connected modulo p ; see [5]. \square

Remark 3.2. One must be careful in the proof above; it is tempting to try to use simply $T'_1(\Delta_{i_0})$ in place of $T'(\Delta_{i_0})$. The problem with this is that $T'_1(\Delta_{i_0})$ is not represented by a path in $\{u_0\} \times \mathring{\mathbb{D}}_\omega - \{(u_0, 0)\}$ and, thus, there is no guaranteed swing isotopy to a corresponding path in $\mathring{\mathbb{D}}_\delta \times \{v_0\}$.

In the corollary below, we obtain a conclusion when the dimension of Σf is arbitrary. We use the notation and terminology from [11]. In particular, $\lambda_{f, \mathbf{z}}^s(\mathbf{0})$ is the s -dimensional Lê number of f at the origin with respect to the coordinates \mathbf{z} .

Corollary 3.3. *Suppose that the dimension of Σf at the origin is s , where $s \geq 1$ is arbitrary. Assume that the coordinates $\mathbf{z} := (z_0, \dots, z_{s-1})$ are prepolar for f at the origin, and that the s -dimensional relative polar variety $\Gamma_{f, \mathbf{z}}^s$ at the origin is not empty.*

Then, both $\text{rank } \tilde{H}_{n-s}(F_f)$ and $\dim \tilde{H}_{n-s}(F_f; \mathbb{Z}/p\mathbb{Z})$ are strictly less than $\lambda_{f, \mathbf{z}}^s(\mathbf{0})$.

Proof. One simply takes the codimension $s - 1$ linear slice $N := V(z_0, \dots, z_{s-2})$ through the origin. Then, $f|_N$ has a 1-dimensional critical locus and, by iterating Theorem 1.1, $\tilde{H}_{n-s}(F_f) \cong \tilde{H}_{(n-s+1)-1}(F_{f|_N})$. Now, by Proposition 1.21 of [11], $\lambda_{f, \mathbf{z}}^s(\mathbf{0}) = \lambda_{f|_N, z_{s-1}}^1(\mathbf{0})$. The corollary now follows at once from Theorem 3.1 (the proof with $\mathbb{Z}/p\mathbb{Z}$ coefficients is identical). \square

As we shall see, Corollary 3.3 puts restrictions on the types of perverse sheaves that one may obtain as vanishing cycles of the shifted constant sheaf on affine space. Below, we refer to the constant sheaf on ν of dimension $\mathring{\mu}_\nu$, shifted by 1 and extended by zero to all of $V(f)$; we write $(k^{\mathring{\mu}_\nu})_\nu^\bullet[1]$ for this sheaf (note that we omit the reference to the extension by zero in the notation). The isomorphisms and direct sums that we write below are in the Abelian category of perverse sheaves.

In the trivial case, Σf consists of a single smooth component ν and $\phi_f[-1]k_{\mathcal{U}}^\bullet[n+1] \cong (k^{\mathring{\mu}_\nu})_\nu^\bullet[1]$. Aside from the trivial case, is it possible for $(k^{\mathring{\mu}_\nu})_\nu^\bullet[1]$ to be a direct summand of $\phi_f[-1]k_{\mathcal{U}}^\bullet[n+1]$? The following corollary provides a partial answer, and generalizes the question/answer to critical loci of arbitrary dimension.

Corollary 3.4. *Suppose that the critical locus of f is s -dimensional, where $s \geq 1$ is arbitrary. For each s -dimensional component ν of Σf , let $\mathring{\mu}_\nu$ denote the Milnor number of f restricted to a generic normal slice of ν .*

If Σf is smooth and the generic s -dimensional relative polar variety of f is empty, then $\phi_f[-1]k_{\mathcal{U}}^\bullet[n+1] \cong (k^{\mathring{\mu}_\nu})_\nu^\bullet[s]$.

If each component of Σf is smooth, and the generic s -dimensional relative polar variety of f is not empty, then $\bigoplus_\nu (k^{\mathring{\mu}_\nu})_\nu^\bullet[s]$ is not a direct summand of $\phi_f[-1]k_{\mathcal{U}}^\bullet[n+1]$.

Proof. If Σf is smooth and the s -dimensional relative polar variety is empty, $V(f)$ has an a_f stratification consisting of two strata: $V(f) - \Sigma f$ and Σf . As $\phi_f[-1]k_{\mathcal{U}}^\bullet[n+1]$ is constructible with respect to any a_f stratification, the first statement follows.

If each component of Σf is smooth, then, for generic coordinates, the s -dimensional Lê number $\lambda_f^s(\mathbf{0})$ will be equal to $\sum_{\nu} \overset{\circ}{\mu}_{\nu}$, where we sum over s -dimensional components. Now, the second statement follows at once from Corollary 3.3, since such a direct summand would immediately imply that the dimension of $\tilde{H}_{n-s}(F_f)$ is too big. \square

4 Comments, Questions, and Counterexamples

One might hope that a stronger result than Theorem 3.1 is true.

For instance, given that Theorem 3.1 and Theorem 1.4 are true, it is natural to ask the following:

Question 4.1. If we are not in the trivial case, is the rank of $\tilde{H}_{n-1}(F_f)$ strictly less than $\sum_{\nu} \overset{\circ}{\mu}_{\nu}$?

The answer to the above question is “no”. One can find examples of this in the literature, but perhaps the easiest is the following:

Example 4.2. Let $f := (y^2 - x^3)^2 + w^2$. Then, Σf has a single component $\nu := V(w, y^2 - x^3)$, and one easily checks that $\overset{\circ}{\mu}_{\nu} = 1$. However, as f is the suspension of $(y^2 - x^3)^2$, the Sebastiani-Thom Theorem (here, we need the version proved by Oka in [12]) implies

$$\tilde{H}_1(F_f) \cong \tilde{H}_0(F_{(y^2-x^3)^2}) \cong \mathbb{Z}.$$

Moreover, by suspending f again, one may produce an example in which f itself has a single irreducible component at the origin.

Now, let α be the number of irreducible components of Σf .

Question 4.3. If we are not in the trivial case, is the rank of $\tilde{H}_{n-1}(F_f)$ strictly less than $\lambda^1 - \alpha$?

Again, there are many examples in the literature which demonstrate that the answer to this question is “no”. One simple example is:

Example 4.4. The function $f = x^2y^2 + w^2$ has a critical locus consisting of two lines, $\lambda^1 = 2$, but – using the Sebastiani-Thom Theorem again – we find that $\tilde{H}_1(F_f) \cong \mathbb{Z}$.

However, a result such as that asked about in Question 4.3, but where α is replaced by a quantity involving the number of components of Γ , or numbers of various types of components in the Cerf diagram, seems more likely. Moreover, if we put more conditions on the intersection diagram for the vanishing cycles of f_0 , we could certainly obtain sharper bounds than we do in the Main Theorem. Or, if we know more

topological data, such as the vertical monodromies, as in [14], we could obtain better bounds. However, other than Theorem 3.1, we know of no nice, effectively calculable, formula which holds in all cases.

Finally, Corollary 3.4 leads us to ask:

Question 4.5. Which perverse sheaves can be obtained as the vanishing cycles of the constant sheaf on affine space?

Unlike our previous questions, we do not know the answer to Question 4.5.

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