

Appendix A: Analytic Cycles and Intersections

We wish to consider schemes, cycles, and sets. Frequently, we will be in the algebraic setting and, hence, we may use algebraic schemes, cycles, and sets. However, as we wish to treat the more general analytic case, we should clarify what we mean by the terms scheme and cycle.

In the analytic setting, by *scheme*, we actually mean a (not necessarily reduced) complex analytic space, (X, \mathcal{O}_X) , in the sense of [G-R1] and [G-R2]. By the irreducible components of X , we mean simply the irreducible components of the underlying analytic set X . If we concentrate our attention on the germ of X at some point \mathbf{p} , then we may discuss embedded subvarieties and (non-embedded, or isolated) components of the germ of X at \mathbf{p} – these correspond to non-minimal and minimal primes, respectively, in the set of associated primes of the Noetherian local ring $\mathcal{O}_{X, \mathbf{p}}$.

If X is a complex space and α is a coherent sheaf of ideals in \mathcal{O}_X , then we write $V(\alpha)$ for the possibly non-reduced analytic subspace defined by the vanishing of α .

By the intersection of a collection of closed subschemes, we mean the scheme defined by the sum of the underlying ideal sheaves. By the union of a finite collection of closed subschemes, we mean the scheme defined by the intersection (not the product) of the underlying ideal sheaves. We say that two subschemes, V and W , are equal up to embedded subvariety provided that, in each stalk, the isolated components of the defining ideals (those corresponding to minimal primes) are equal. Our main concern with this last notion is that it implies that the cycles $[V]$ and $[W]$ are equal (see below).

A.1 Given an analytic space X (with its reduced structure), an *analytic cycle* in X is a formal sum $\sum m_V [V]$, where the V 's are irreducible analytic subsets of X , the m_V 's are integers, and the collection $\{V\}$ is a locally finite collection of subsets of X . As a cycle is a locally finite sum, and as we will normally be concentrating on the germ of an analytic space at a point, usually we can safely assume that a cycle is actually a finite formal sum.

Throughout this book, whenever we write a cycle $\sum m_V [V]$, we shall assume that the V 's are distinct and that none of the m_V 's are zero. This is the same as saying that the presentation is minimal, in the sense that no further cancellations are possible.

We say that a cycle $\sum m_V [V]$ is *positive* if $m_V > 0$ for all V ; a cycle is *non-negative* if it is the zero-cycle or is positive.

A.2 Given an analytic space, (X, \mathcal{O}_X) , we wish to define the (positive) cycle associated to (X, \mathcal{O}_X) . In the algebraic context, this is given by Fulton in [Fu, 1.5] as

$$[X] := \sum m_V [V],$$

where the V 's run over all the irreducible components of X , and m_V equals the length of the ring $\mathcal{O}_{X, V}$, the local ring of X along V . In the analytic context, we wish to use the same definition, but we must be more careful in defining the m_V .

Define m_V as follows. Take a point \mathbf{p} in V . The germ of V at \mathbf{p} breaks up into irreducible germ components $(V_{\mathbf{p}})_i$. Take any one of the $(V_{\mathbf{p}})_i$ and let m_V equal the length of the ring $(\mathcal{O}_{X, \mathbf{p}})_{(V_{\mathbf{p}})_i}$ (that is, the local ring of X at \mathbf{p} localized at the prime corresponding to $(V_{\mathbf{p}})_i$). This number is independent of the point \mathbf{p} in V and the choice of $(V_{\mathbf{p}})_i$.

Note that any embedded subvarieties of a scheme do **not** contribute to the associated cycle.

One can easily show that, if $f, g \in \mathcal{O}_x$, then $[V(fg)] = [V(f)] + [V(g)]$; in particular, $[V(f^m)] = m[V(f)]$.

If Y is an analytic subset of X and C is a cycle in Y , then we may naturally consider C as a cycle in X .

We shall be dealing with analytic schemes, cycles, and analytic sets. For clarification of what structure we are considering, we shall at times enclose cycles in square brackets, $[]$, and analytic sets in a pair of vertical lines, $||$. Occasionally, when the notation becomes cumbersome, we shall simply state explicitly whether we are considering V as a scheme, a cycle, or a set.

We say that two cycles are equal at a point, \mathbf{p} , provided that the portions of each cycle which pass through \mathbf{p} are equal. When we say that a space, X , is purely k -dimensional at a point, \mathbf{p} , we mean to allow for the vacuous case where X has no components through \mathbf{p} .

We wish to describe some aspects of intersection theory. Of course, [Fu] is the definitive reference for this subject. However, we deal only with cycles, **not** cycle classes, and we deal only with **proper** intersections inside complex **manifolds**; this makes much of the theory fairly trivial to describe.

A.3 If V and W are irreducible subschemes of a connected complex manifold, M , and Z is an irreducible component of $V \cap W$ such that $\text{codim}_M Z = \text{codim}_M V + \text{codim}_M W$, then we say that V and W *intersect properly along Z* , or that Z is a *proper component of $V \cap W$* . Two irreducible subschemes V and W in a connected complex manifold, M , are said to *intersect properly* in M provided that they intersect properly along each component of $V \cap W$; when this is the case, the *intersection product* of $[V]$ and $[W]$ in M is characterized axiomatically by four properties listed below: openness, transversality, projection, and continuity (see [Fu], Example 11.4.4).

A.4 If α is a coherent sheaf of ideals in \mathcal{O}_M and $f \in \mathcal{O}_M$ is such that $V(f)$ contains no embedded subvarieties or irreducible components of $V(\alpha)$, then $V(\alpha)$ and $V(f)$ intersect properly in M and $[V(\alpha)] \cdot [V(f)] = [V(\alpha + \langle f \rangle)]$ (see [Fu], 7.1.b). This statement immediately implies one which, a priori, seems stronger: if α and f are as before and $V(f)$ contains no irreducible component of $V(\alpha)$ and contains no embedded subvariety **which is of codimension one inside some irreducible component of $V(\alpha)$** , then $V(\alpha)$ and $V(f)$ intersect properly in M and $[V(\alpha)] \cdot [V(f)] = [V(\alpha + \langle f \rangle)]$.

More generally, if $W := V(\alpha)$ is a subscheme of M and $f_1, \dots, f_k \in \mathcal{O}_M$ determine regular sequences in the stalks $\mathcal{O}_{W, \mathbf{p}}$ and $\mathcal{O}_{M, \mathbf{p}}$ at all points $\mathbf{p} \in W \cap V(f_1, \dots, f_k)$, then

$$[V(\alpha)] \cdot [V(f_1, \dots, f_k)] = [V(\alpha + \langle f_1, \dots, f_k \rangle)].$$

The two paragraphs above allow one to define the intersection with a hypersurface (or, more generally, a Cartier divisor) without having to refer to an ambient manifold. Suppose that $V(\alpha)$ is a subscheme of an analytic space X , that X is contained in an analytic manifold M , and that $f \in \mathcal{O}_X$. Then, locally, $\mathcal{O}_X \cong \mathcal{O}_M/\gamma$ for some coherent sheaf of ideals $\gamma \subseteq \mathcal{O}_M$. Let $\tilde{\alpha} \subseteq \mathcal{O}_M$ be a coherent sheaf of ideals such that $\gamma \subseteq \tilde{\alpha}$ and such that $\tilde{\alpha}/\gamma$ corresponds to α , i.e., $\tilde{\alpha}$ is such that $V(\alpha) = V(\tilde{\alpha})$. Let \tilde{f} be an extension of f to M . If $V(f)$ contains no embedded subvarieties or isolated components of $V(\alpha)$, then $V(\tilde{f})$ contains no embedded subvarieties or isolated components of $V(\tilde{\alpha})$ and so, by the previous paragraph, $([V(\tilde{\alpha})] \cdot [V(\tilde{f})]; M) = [V(\tilde{\alpha} + \langle \tilde{f} \rangle)]$, which defines the same cycle in X as does $[V(\alpha + \langle f \rangle)]$. Therefore, by locality, we may unambiguously define $([V(\alpha)] \cdot [V(f)]; X)$ by setting it equal to $[V(\alpha + \langle f \rangle)]$.

A.5 Two cycles $\sum m_i[V_i]$ and $\sum n_j[W_j]$ are said to intersect properly if V_i and W_j intersect properly for all i and j ; when this is the case, the intersection product is extended bilinearly by defining

$$\sum m_i[V_i] \cdot \sum n_j[W_j] = \sum m_i n_j ([V_i] \cdot [W_j]).$$

Occasionally it is useful to include the ambient manifold in the notation; in these cases we write $(C_1 \cdot C_2; M)$ for the proper intersection of cycles C_1 and C_2 in M .

If two cycles C_1 and C_2 intersect properly and $C_1 \cdot C_2 = \sum n_k[Z_k]$, where the Z_k are irreducible, then the *intersection number* of C_1 and C_2 at Z_k , $(C_1 \cdot C_2)_{Z_k}$, is defined to be n_k ; that is, the number of times Z_k occurs in the intersection, counted with multiplicity. Note that, when C_1 and C_2 have complementary codimensions, all the Z_k are merely points.

A.6 It is fundamental that $(C_1 \cdot C_2)_{Z_k}$ can be calculated locally; that is, if \mathcal{U} is an open subset of M such that $Z_k \cap \mathcal{U} \neq \emptyset$, then

$$\text{(openness)} \quad (C_1 \cap \mathcal{U} \cdot C_2 \cap \mathcal{U}; \mathcal{U})_{Z_k \cap \mathcal{U}} = (C_1 \cdot C_2; M)_{Z_k}$$

(see [Fu], 11.4.4).

A.7 If V and W are two irreducible subvarieties of M and P is an irreducible component of $V \cap W$, we say that V and W are *generically transverse along P in M* provided that V and W are reduced and, at generic points of P , V and W are smooth and intersect transversely in M ; naturally, we say that V and W are *generically transverse in M* provided they are generically transverse along every component of the intersection. Another fundamental property of intersection numbers is the **transversality characterization**:

if V and W are irreducible subschemes of M which intersect properly along an irreducible component P , then $(V \cdot W)_P = 1$ if and only if V and W are generically transverse along P in M ([Fu], 8.2.c and 11.4.4).

A.8 If C_1, C_2 , and C_3 are positive cycles such that C_1 and C_2 intersect properly, and C_3 properly intersects $C_1 \cdot C_2$, then C_2 and C_3 intersect properly, C_1 properly intersects $C_2 \cdot C_3$, and

$$\text{(associativity)} \quad (C_1 \cdot C_2) \cdot C_3 = C_1 \cdot (C_2 \cdot C_3).$$

We wish to introduce a slight generalization of proper intersections of cycles. If V and W are irreducible subschemes of a connected complex manifold, M , and Z is an irreducible component of $V \cap W$ along which V and W intersect properly, then, for every open neighborhood $\mathcal{U} \subseteq M$ such that $V \cap W \cap \mathcal{U} = Z \cap \mathcal{U}$, the value of $(V \cap \mathcal{U} \cdot W \cap \mathcal{U}; \mathcal{U})_{Z \cap \mathcal{U}}$ is independent of \mathcal{U} ; we define $(V \cdot W)_Z$ to be this common value. If Z is a proper component of $V \cap W$, then, by [Fu], 8.2.a, $(V \cdot W)_Z \leq [V \cap W]_Z$.

We define $[V] \cdot_p [W] := \sum_Z (V \cdot W)_Z$, where the sum is over all Z along which V and W intersect properly. We extend bilinearly

$$\sum m_i[V_i] \cdot_p \sum n_j[W_j] = \sum m_i n_j ([V_i] \cdot_p [W_j]).$$

We refer to this as *the proper intersection* of the two cycles.

One easily verifies that, if C_1, C_2 , and C_3 are positive cycles, then

$$(C_1 \cdot_p C_2) \cdot_p C_3 = C_1 \cdot_p (C_2 \cdot_p C_3).$$

A.9 Given a point $\mathbf{p} \in M$, a curve $W = V(\alpha)$ in M which is reduced and irreducible at \mathbf{p} , and a hypersurface $V(f) \subseteq M$ which intersects W properly at \mathbf{p} , there is a very useful way to calculate the intersection number $([W] \cdot [V(f)])_{\mathbf{p}}$. One takes a local parameterization $\phi(t)$ of W which takes 0 to \mathbf{p} , and then $([W] \cdot [V(f)])_{\mathbf{p}} = \text{mult}_t f(\phi(t))$, the degree of the lowest non-zero term. This is easy to see, for composition with ϕ induces an isomorphism

$$\frac{\mathcal{O}_{M,\mathbf{p}}}{\alpha + \langle f \rangle} \xrightarrow{\circ\phi} \frac{\mathbb{C}\{t\}}{f(\phi(t))}.$$

Of course, if c is small and unequal to zero, $\text{mult}_t f(\phi(t))$ is precisely the number of roots of $f(\phi(t)) - c$ which occur near zero.

A.10 Combining this with transversality, we obtain the following *dynamic intersection property*:

$$(V(\alpha) \cdot V(f))_{\mathbf{p}} = \sum_{\mathbf{q} \in \mathring{B}_\epsilon \cap V(\alpha) \cap V(f-c)} (V(\alpha) \cdot V(f-c))_{\mathbf{q}},$$

where $\epsilon > 0$ is sufficiently small, \mathring{B}_ϵ is an open ball of radius ϵ centered at \mathbf{p} , and $|c| \ll \epsilon$.

This formula may seem ridiculously complex, since all the $(V(\alpha) \cdot V(f-c))_{\mathbf{q}}$ equal 1; however, it is the form which generalizes nicely: if C is a purely one-dimensional cycle and $V(f)$ properly intersects $|C|$ at \mathbf{p} , then

$$(C \cdot V(f))_{\mathbf{p}} = \sum_{\mathbf{q} \in \mathring{B}_\epsilon \cap |C| \cap V(f-c)} (C \cdot V(f-c))_{\mathbf{q}}.$$

This is a special case of *conservation of number*, which we shall discuss more generally below.

A.11 The *projection formula* ([Fu], 11.4.4.iii) allows us to calculate intersections inside normal slices. Let $C = \sum n_i [V_i]$ be a cycle in M and let N be a closed submanifold of M such that N generically transversely intersects each V_i in M (this is equivalent to: for each component Z of $V_i \cap N$, $(V_i \cdot N)_Z = 1$).

We may consider $(C \cdot N; M)$ as a cycle in N ; denote this cycle by \tilde{C} . Let B be a cycle in N ; we may also consider B as a cycle in M . Then, the projection formula states:

$$\text{(projection formula)} \quad (\tilde{C} \cdot_p B; N) = (C \cdot_p B; M).$$

The projection formula lets us reduce the problem of calculating intersection numbers to the case where the intersection consists of isolated points. To see this, suppose that C_1 and C_2 are two cycles in M which intersect properly and let $C_1 \cdot C_2 = \sum n_i [V_i]$. To calculate n_{i_0} , first let \mathbf{p} be a smooth point of V_{i_0} which is not contained in any other V_i of dimension less than or equal to that of V_{i_0} . Now, take a *normal slice*, N , to V_{i_0} at a smooth point, \mathbf{p} , of V_{i_0} ; that is, in an open neighborhood, \mathcal{U} , of \mathbf{p} in M , N is a closed submanifold of \mathcal{U} of complementary codimension to V_{i_0} such that N transversely intersects V_{i_0} inside \mathcal{U} in the single point \mathbf{p} and such that N is generically transverse to all other V_i and to all components of C_1 and C_2 in \mathcal{U} . By locality $n_{i_0} = (C_1 \cap \mathcal{U} \cdot C_2 \cap \mathcal{U}; \mathcal{U})_{V_{i_0} \cap \mathcal{U}}$. As $V_{i_0} \cap \mathcal{U}$ is the only component of $(C_1 \cap \mathcal{U} \cdot C_2 \cap \mathcal{U}; \mathcal{U})$ whose intersection with N gives $\{\mathbf{p}\}$, the transversality characterization yields that $n_{i_0} = (C_1 \cap \mathcal{U} \cdot C_2 \cap \mathcal{U} \cdot N; \mathcal{U})_{\mathbf{p}}$. But, we wish to calculate this intersection inside of the normal slice N – this is what we get from the projection formula.

Replace the M , N , C , and B in the projection formula as stated above by letting $M = \mathcal{U}$, $N = N$, $C = C_1 \cap \mathcal{U}$, and $B = (C_2 \cap \mathcal{U}) \cdot N$ (consider B as a cycle in N). Then, the formula yields that

$$((C_1 \cap \mathcal{U}) \cdot N) \cdot ((C_2 \cap \mathcal{U}) \cdot N); N = (C_1 \cap \mathcal{U} \cdot C_2 \cap \mathcal{U} \cdot N; \mathcal{U}).$$

Thus, we see that taking normal slices reduces calculating proper intersections of cycles to the case where the dimension of the intersection is zero.

A.12 There is one last property of intersections of cycles that we need – continuity ([Fu], 11.4.4.iii). This property is what makes intersections dynamic; one can move the intersections in a family.

Let M be a analytic manifold and let $\mathring{\mathbb{D}}$ be an open disc centered at the origin in \mathbb{C} . Then, the projection $M \times \mathring{\mathbb{D}} \xrightarrow{\pi} \mathring{\mathbb{D}}$ determines a one-parameter family of spaces, and any subscheme $W \subseteq M \times \mathring{\mathbb{D}}$ determines a one-parameter family of schemes $W_t := W \cap (M \times \{t\})$. Hence, any cycle $C := \sum n_i[V_i]$ in $M \times \mathring{\mathbb{D}}$ determines a family of cycles $C_t := \sum n_i[(V_i)_t]$ in $M \cong M \times \{t\}$.

If a cycle C in $M \times \mathring{\mathbb{D}}$ has a component contained in $M \times \{t\}$ for some t , then that component does not “propagate” through the family; we wish to eliminate such “bad” components. For any cycle $C = \sum n_i[V_i]$ in $M \times \mathring{\mathbb{D}}$ and any analytic set $W \subseteq M \times \mathring{\mathbb{D}}$, let

$$C \dashv W := \sum_{V_i \not\subseteq W} n_i[V_i],$$

and let

$$C_t^* := \left((C \dashv (M \times \{t\})) \cdot (M \times \{t\}); M \times \mathring{\mathbb{D}} \right) = \sum_{V_i \not\subseteq M \times \{t\}} n_i[(V_i)_t].$$

Continuity of intersections states that, if C is a cycle in $M \times \mathring{\mathbb{D}}$ with no component contained in $M \times \{0\}$, and E is a cycle in M such that C_0 properly intersects E in M , then there exists a (possibly) smaller disk centered at the origin $\mathring{\mathbb{D}}' \subseteq \mathring{\mathbb{D}}$ such that C properly intersects $E \times \mathring{\mathbb{D}}'$ in $M \times \mathring{\mathbb{D}}'$ and, for all $t \in \mathring{\mathbb{D}}'$,

$$\text{(continuity)} \quad \left((E \times \mathring{\mathbb{D}}') \cdot C; M \times \mathring{\mathbb{D}}' \right)_t^* = (E \cdot C_t; M).$$

A.13 We saw earlier how the projection formula allows us to reduce the calculation of intersection multiplicities to the case where the intersection is zero-dimensional. We wish to see now how continuity allows us to deform in a family in order to calculate zero-dimensional intersection multiplicities.

We will prove a dynamic formula for intersection multiplicities; this formula is known as **conservation of number**. Let E be a k -dimensional cycle in M and let $\mathbf{f} := (f_1, \dots, f_k) \in (\mathcal{O}_M)^k$ be such that E and $V(\mathbf{f})$ intersect properly in the single point \mathbf{p} . This implies that $V(\mathbf{f})$ is purely k -codimensional inside M at \mathbf{p} . In what follows, we assume that we are always working in an arbitrarily small neighborhood of \mathbf{p} .

Let $g_1(\mathbf{z}, t), \dots, g_k(\mathbf{z}, t) \in \mathcal{O}_{M \times \mathring{\mathbb{D}}}$ be such that $g_i(\mathbf{z}, 0) = f_i(\mathbf{z})$ for all i . Let C be the cycle in $M \times \mathring{\mathbb{D}}$ given by $[V(g_1(\mathbf{z}, t), \dots, g_k(\mathbf{z}, t))]$. Note that $C_0 = [V(\mathbf{f})]$ (in M) and that C has no

components contained in $M \times \{0\}$, for otherwise $V(\mathbf{f})$ would have a component of dimension at least $(\dim M) + 1 - k$.

Applying continuity at $t = 0$, we find that

$$(E \cdot V(\mathbf{f}); M) = \left((E \times \mathring{\mathbb{D}}') \cdot C; M \times \mathring{\mathbb{D}}' \right)_0^*$$

for a smaller disc $\mathring{\mathbb{D}}'$. Note that $(E \times \mathring{\mathbb{D}}') \cdot C$ is a purely 1-dimensional cycle, say $\sum_j m_j [W_j]$. Applying continuity at general t , we find that, for all $t \in \mathring{\mathbb{D}}'$,

$$(E \cdot C_t; M) = \sum_{W_j \not\subseteq M \times \{0\}} m_j [(W_j)_t].$$

Hence, by openness and transversality, we find that $m_j = (E \cdot C_t; M)_{\mathbf{q}}$ for sufficiently small $t \neq 0$ and $\mathbf{q} \in (W_j)_t$. Now, since $(E \cdot V(\mathbf{f}); M) = \sum_{W_j \not\subseteq M \times \{0\}} m_j [(W_j)_0]$, we may apply our earlier special case of dynamic intersections between curves and hypersurfaces to conclude the general **conservation of number** formula

$$(E \cdot V(\mathbf{f}))_{\mathbf{p}} = \sum_{\mathbf{q} \in \mathring{B}_\epsilon \cap |E \cap C_t|} (E \cdot C_t)_{\mathbf{q}},$$

where $\epsilon > 0$ is sufficiently small, \mathring{B}_ϵ is an open ball of radius ϵ centered at \mathbf{p} , $|t| \ll \epsilon$, and C_t equals $[V(g_1(\mathbf{z}, t), \dots, g_k(\mathbf{z}, t))]$.

A.14 Finally, we need to define the *proper push-forward* of cycles (see [Fu, 1.4]). Let $f : X \rightarrow Y$ be a proper morphism of analytic spaces. Then, for each irreducible subvariety $V \subseteq X$, $W := f(V)$ is an irreducible subvariety of Y . There is an induced embedding of rational function fields $R(W) \hookrightarrow R(V)$, which is a finite field extension if V and W have the same dimension. Define the *degree of V over W* by

$$\deg(V/W) := \begin{cases} [R(V) : R(W)] & \text{if } \dim W = \dim V \\ 0 & \text{if } \dim W \neq \dim V, \end{cases}$$

where $[R(V) : R(W)]$ denotes the degree of the field extension, which equals the number of points in $V \cap f^{-1}(\mathbf{p})$ for a generic choice of $\mathbf{p} \in W$.

Define $f_*(V)$ by

$$f_*(V) = \deg(V/W)[W].$$

This extends linearly to a homomorphism which is called the proper push-forward of cycles:

$$f_* \left(\sum m_V [V] \right) = \sum m_V f_*(V).$$

We will need the following special case of the more general *push-forward formula* (see [Fu, 2.3.c]). Let $\pi : M \rightarrow N$ be a proper map between analytic manifolds. Let $f \in \mathcal{O}_N$. Let C be a cycle in M which intersects $V(f \circ \pi)$ properly in M . Then, $V(f)$ properly intersects $\pi_*(C)$ in N and

$$\text{(push-forward formula)} \quad \pi_*(V(f \circ \pi) \cdot C) = V(f) \cdot \pi_*(C).$$

We need one other formula involving the push-forward and graphs of morphisms. Suppose that we have an analytic map $f : M \rightarrow N$ between analytic manifolds. Then, for any irreducible subvariety $V \subseteq M$, the graph of $f|_V$, $\text{Gr}(f|_V)$ is isomorphic to V . Thus, one would expect that intersecting with V in M could be identified with intersecting with $\text{Gr}(f|_V)$ in $M \times N$; this is, in fact the case.

Suppose that $A := \sum m_i[V_i]$ and B are properly intersecting cycles in M . Let $\text{Gr}(A) := \sum m_i[\text{Gr}(f|_{V_i})]$ in $M \times N$, and let $\text{pr} : M \times N \rightarrow M$ denote the projection. Then, $\text{Gr}(A)$ properly intersects $B \times N$ in $M \times N$, and we have the *graph formula*:

$$\text{(graph formula)} \quad (A \cdot B; M) = \text{pr}_*(\text{Gr}(A) \cdot (B \times N); M \times N).$$

This is easy to see: $\text{Gr}(A) = (A \times N) \cdot \text{Gr}(f)$, since $\text{Gr}(f)$ determines a regular sequence in each $\mathcal{O}_{V_i \times N}$ (see A.4). Hence,

$$\text{Gr}(A) \cdot (B \times N) = (A \times N) \cdot (B \times N) \cdot \text{Gr}(f) = ((A \cdot B) \times N) \cdot \text{Gr}(f),$$

where the last equality follows from normal slicing (see A.11). The graph formula follows easily by using A.4 again, together with the definition of the proper push-forward.

