A LITTLE MICROLOCAL MORSE THEORY

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Abstract. If a complex analytic function, f, has a stratified isolated critical point, then it is known that the cohomology of the Milnor fibre of f has a direct sum decomposition in terms of the normal Morse data to the strata. We use microlocal Morse theory to obtain the same result under the weakened hypothesis that the vanishing cycles along f have isolated support. We also investigate an index-theoretic proof of this fact.

§0. Introduction

Our goal in this paper is very modest. Let X be a complex analytic space, let $\mathcal{S} := \{S_\alpha\}$ be a Whitney stratification of X with connected strata, let $f : X \to \mathbb{C}$ be a complex analytic function, let R be a principal ideal domain (p.i.d.), and let $\mathbf{F}^\bullet$ be a bounded complex of sheaves of R-modules on X, which is constructible with respect to $\mathcal{S}$. (Typically, one choses the base ring R to be $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{C}$.) We wish to show: if $\mathbf{x}$ is an isolated point in the support of the vanishing cycles $\phi_f\mathbf{F}^\bullet$, then the stalk cohomology of $\phi_f\mathbf{F}^\bullet$ at $\mathbf{x}$ is a direct sum, with multiplicities, of the $\mathbf{F}^\bullet$-Morse modules of strata of X (i.e., hypercohomologies of normal data of strata, with coefficients in $\mathbf{F}^\bullet$).

As we are interested in activity at an isolated point, our question is entirely local. Hence, there is no loss of generality in assuming that X is embedded in an open subset, $\mathcal{U}$, of some affine space, that $\mathbf{x} = \mathbf{0}$, and that $\hat{f}$ is an analytic extension of f to all of $\mathcal{U}$.

To state our result precisely, we need to introduce terminology that will appear in Definition 3.3: a stratum S_α is $\mathbf{F}^\bullet$*-visible* if its $\mathbf{F}^\bullet$-Morse modules are not all trivial. In Section 5, we prove:

Main Theorem (5.3). *Suppose that $\mathbf{0}$ is an isolated point in the support of $\phi_f\mathbf{F}^\bullet$.*

Then, for every $\mathbf{F}^\bullet$-visible stratum S_α, $(\mathbf{0}, d_\mathbf{0}\hat{f})$ is an isolated point in $\overline{T^*_{S_\alpha}\mathcal{U}} \cap \operatorname{im} d\hat{f}$, and if we let k_α equal the intersection multiplicity $\left(\overline{T^*_{S_\alpha}\mathcal{U}} \cdot \operatorname{im} d\hat{f}\right)_{(\mathbf{0}, d_\mathbf{0}\hat{f})}$, then there is a (non-canonical) isomorphism

$$H^{i-1}(\phi_f\mathbf{F}^\bullet)_\mathbf{0} \cong \bigoplus_{\substack{\mathbf{F}^\bullet\text{-}visible\\ S_\alpha}} \left(R^{k_\alpha} \otimes_R \mathbb{H}^{i-d_\alpha}(\mathbb{N}_\alpha, \mathbb{L}_\alpha\ ;\ \mathbf{F}^\bullet)\right).$$

The knowledgeable reader may believe that this result is already known; on the cohomological level, it is a generalization of the fundamental theorem of stratified Morse Theory: that local Morse data is the product of tangential and normal Morse data. Moreover, there are similar results of Siersma [**Si**], Tibăr [**Ti**], and in our own work in [**M2**]. However, in each of these papers, the hypothesis is that f has a **stratified** isolated critical point $\mathbf{x}$. While this assumption certainly implies that $\mathbf{x}$ is an isolated point in $\operatorname{supp}\phi_f\mathbf{F}^\bullet$, it is easily seen to be a strictly stronger condition. From a philosophical point of view, one would like to be able to conclude something about the structure of $\phi_f\mathbf{F}^\bullet$ from a condition on $\operatorname{supp}\phi_f\mathbf{F}^\bullet$.

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We first prove, in Section 2, a weak version of the main theorem; we accomplish this by using a vanishing index theorem, proved independently by Lê [**L1**], Sabbah [**Sa**], and Ginsburg [**G**], combined with the tool of perverse cohomology (see [**BBD**] and [**K-S3**]). Perverse cohomology allows us to take this index result about the Euler characteristic and use it to extract the individual Betti numbers. This proof is very short. However, it has the disadvantage of being too magical; in no way does one "see" the contributions from the critical points of a small perturbation of f.

As a consequence of its mystical nature, the proof does not generalize to other situations where one might hope to improve Morse-theoretic results. Moreover, one can not quite recover the integral cohomology by this method. Hence, we wish to go back to the Morse theory proof and try to make it work.

How difficult is it to pass from the Morse theory proof in the stratified isolated case to the case of isolated support for the vanishing cycles? Fairly, and the difficulties are subtle. Essentially, the problem is to show that $\mathbf{F}^\bullet$-invisible strata are not relevant. Proving that such invisible strata do not matter either requires some very tedious arguments about types of neighborhoods that one may use to calculate the normal data at points on the boundary of a small closed ball, or requires one to use the microlocal theory developed by Kashiwara and Schapira in [**K-S1**], [**K-S2**], and [**K-S3**].

Kashiwara and Schapira indicate ([**K-S3**], Chapt. V Notes) that the micro-support can be viewed as a generalization of stratified Morse theory – for it applies even to complexes of sheaves which are not constructible. In this paper, we wish to expound on two further advantages of the microlocal approach. The first is that, by definition, the micro-support ignores superfluous strata. The second is that the general machinery of the microlocal theory enables one to circumvent many Morse theoretic arguments involving choices of special kinds of neighborhoods.

However, we still find the language and ideas of stratified Morse theory to be very intuitive. Thus, our general approach is a mixture of stratified Morse theory and the microlocal theory – we think in terms of Morse theory, but write down the microlocal proofs.

A brief outline of this paper is as follows.

In Section 1, we establish some general notation and terminology that we shall use throughout the rest of the paper. Despite the fact that the main theorem (5.3) of this paper is a statement in the complex analytic setting, we will need to work in the real subanalytic category; hence, our notation must be more general than that which we have used in the introduction. Also in Section 1, we state and sketch the proof of our prior result from [**M2**]; we do this to provide background and because the proof of the main theorem of this paper follows along the same lines, once we have explained the requisite microlocal theory.

In Section 2, we provide an index-theoretic proof of the main theorem in the case where the base ring is a field. In other words, we show that there is an equality of Betti numbers. This proof is short, but not very geometrically enlightening. Moreover, even switching the base ring to finite fields does not enable one to recover the main theorem with integral coefficients.

In Section 3, we begin our discussion of the technical details necessary to turn the sketch of a proof in Section 1 into a rigorous proof of Theorem 5.3. We discuss switching base rings, $\mathbf{F}^\bullet$-visible strata, and the support of the vanishing cycles. The fundamental results of this section are Proposition 3.2 (Continuity of Vanishing Support) and Theorem 3.4, which provides a conormal characterization of the support of the vanishing cycles.

In Section 4, we summarize the basics of the microlocal theory that we shall need. For the proofs of most of the results of this section, the reader is referred to [**K-S3**]. However, for lack of any convenient reference, we provide a proofs of Theorem 4.13 and Corollary 4.15; these results relate the micro-support of a complex of sheaves to visible strata and the vanishing cycles.

Finally, in Section 5, we use the microlocal theory of Section 4 to turn our earlier proof, sketched in Section 1, into a proof of the main theorem.

§1. **Notation and a Known Result**

In this section, we will fix some notation from stratified Morse theory with coefficients in complexes of sheaves (see [**G-M2**], 6.A), and then describe our result from [**M2**], Theorem 3.2, which is based on the argument in [**Si**]. We also provide a quick sketch of the proof; the point of describing this proof is that the microlocal argument proceeds along the same lines.

Let R be a regular Noetherian ring with finite Krull dimension (e.g., $\mathbb{Z}, \mathbb{Z}/p,\ \mathbb{Q}$, or $\mathbb{C}$). Let X be a real subanalytic set, and let $D^b(X)$ denote the derived category of bounded complexes of R-modules on X. Let $\mathbf{F}^\bullet \in D^b(X)$. If there exists a real subanalytic Whitney stratification, $\mathcal{S}$, of X with respect to which $\mathbf{F}^\bullet$ is constructible, then we write $\mathbf{F}^\bullet \in D^b_{\mathcal{S}}(X)$ and say that $\mathbf{F}^\bullet$ is $\mathbb{R}$-constructible; note that our assumptions about R guarantee that such an $\mathbf{F}^\bullet$ is perfect (see [**G-M1**], 1.4 and [**K-S**], 8.4.3). We denote the full subcategory of $D^b(X)$ of $\mathbb{R}$-constructible complexes by $D^b_{\mathbb{R}}(X)$.

If X is complex analytic, $\mathcal{S}$ is a complex analytic Whitney stratification of X, and $\mathbf{F}^\bullet \in D^b_{\mathcal{S}}(X)$, then we say that $\mathbf{F}^\bullet$ is $\mathbb{C}$-constructible. We denote the full subcategory of $D^b(X)$ of $\mathbb{C}$-constructible complexes by $D^b_{\mathbb{C}}(X)$.

Now suppose that X is a complex analytic set, $\mathcal{S}$ is a complex analytic Whitney stratification of X, and $\mathbf{F}^\bullet \in D^b_{\mathcal{S}}(X)$. For each stratum $S_\alpha \in \mathcal{S}$, there exists *normal data to* S_α, which is a pair $(\mathbb{N}_\alpha, \mathbb{L}_\alpha)$ consisting of a normal slice and the complex link of S_α to S_α; the hypercohomology modules $\mathbb{H}^*(\mathbb{N}_\alpha, \mathbb{L}_\alpha;\ \mathbf{F}^\bullet)$ are well-defined, and we refer to them as the $\mathbf{F}^\bullet$-*Morse modules of* S_α.

Theorem 3.2 of [**M2**] is

Theorem 1.1. *Let X be a complex analytic space embedded in an open subset $\mathcal{U}$ of complex affine space, let $\mathcal{S} := \{S_\alpha\}$ be a complex Whitney stratification of X with connected strata, let $\mathbf{F}^\bullet \in D^b_{\mathcal{S}}(X)$, and let $\hat{f} : \mathcal{U} \to \mathbb{C}$ be complex analytic. Let $f := \hat{f}_{|_X}$, and suppose that $f : (X, \mathbf{0}) \to (\mathbb{C}, 0)$ has a stratified isolated critical point at $\mathbf{0}$.*

Then, there exists a unique set, $\{k_\alpha\}$, of non-negative integers such that, for all bounded complexes, $\mathbf{F}^\bullet$, of R-modules on X which are constructible with respect to $\mathcal{S}$, and, for all i,

$$H^{i-1}(\phi_f\mathbf{F}^\bullet)_\mathbf{0} \ \cong\ \mathbb{H}^i(B_\epsilon \cap X, F_{f,\mathbf{0}}\ ;\mathbf{F}^\bullet) \ \cong\ \mathbb{H}^i(B_\epsilon \cap X \cap f^{-1}(\mathring{\mathbb{D}}_\eta), B_\epsilon \cap X \cap f^{-1}(v)\ ;\mathbf{F}^\bullet)$$

$$\cong\ \bigoplus_\alpha \big(R^{k_\alpha} \otimes_R \mathbb{H}^{i-d_\alpha}(\mathbb{N}_\alpha, \mathbb{L}_\alpha\ ;\ \mathbf{F}^\bullet)\big).,$$

where:

$d_\alpha = \dim S_\alpha$*;*

B_ϵ *is a sufficiently small closed ball of radius ϵ centered at the origin in the ambient affine space;*

$\mathring{\mathbb{D}}_\eta$ *is an open disc of radius η, $0 < \eta \ll \epsilon$, centered at the origin;*

$v \in \mathring{\mathbb{D}}^*_\eta$*;*

$F_{f,\mathbf{0}}$ is the Milnor fibre of f at $\mathbf{0}$ or, more precisely, $F_{f,\mathbf{0}} = B_\epsilon \cap X \cap f^{-1}(v)$.

Moreover, $k_\alpha = 1$ if $S_\alpha = \{\mathbf{0}\}$ and, if $S_\alpha \neq \{\mathbf{0}\}$, then for a generic choice of complex linear forms, L, on the ambient affine space, for all strata S_α, the relative polar curve $\Gamma_{f,L}(S_\alpha)$ is one-dimensional (or empty) at the origin and

$$k_\alpha = \left(\Gamma^1_{f,L}(S_\alpha) \cdot V(f)\right)_{\mathbf{0}} - \left(\Gamma^1_{f,L}(S_\alpha) \cdot V(L)\right)_{\mathbf{0}} \; = \; \left(\overline{T^*_{S_\alpha}\mathcal{U}} \; \cdot \; \operatorname{im} d\hat{f}\right)_{(\mathbf{0}, d_{\mathbf{0}}\hat{f})}.$$

The integer k_α is precisely the number of non-degenerate critical points of a small perturbation of f by L which occur near the origin on the stratum S_α ; more precisely, for all sufficiently small $\delta > 0$, for all complex t such that $0 < |t| \ll \delta$, k_α equals the number of critical points of $f + tL$ in $\overset{\circ}{B}_\epsilon \cap S_\alpha$.

Sketch of the proof:

Fix a small ball B_ϵ. Since f has a stratified isolated critical point at the origin, a small perturbation of f, $g := f + tL$, will have a finite number of isolated Morse stratified critical points in the interior of B_ϵ and no critical points on the boundary. Morover, the Milnor fibre $B_\epsilon \cap V(f - v)$ will be stratified homeomorphic to $B_\epsilon \cap V(g - v)$. Now, apply stratified Morse theory to the function $h := ||g - v||^2$; $h^{-1}(0)$ begins at the Milnor fibre and grows out to having hypercohomology equal to the stalk cohomology of $\mathbf{F}^\bullet$ at the origin. The cohomology of the level sets of h "jumps" by precisely $\mathbb{H}^*(\mathbb{N}_\alpha, \mathbb{L}_\alpha;\ \mathbf{F}^\bullet)$ at each critical point on S_α. The proof would be finished except that one has to prove that there is no cancellation among the contributions from the various critical points.

To show that we get the direct sum decomposition in the theorem, one considers the stratified critical **values** of g, which can be assumed to be distinct. Thus, one can view the whole situation "downstairs" in a small complex disk, $\overset{\circ}{\mathbb{D}}_\eta$, around the origin; one looks at the inverse image of this disk by g, modulo the inverse image of v, and considers what happens at the finite set of critical values. By homotoping and excising, one shows that $\mathbb{H}^*\big(g^{-1}(\overset{\circ}{\mathbb{D}}_\eta), g^{-1}(v);\ \mathbf{F}^\bullet\big)$ breaks up as a direct sum of the hypercohomologies of the inverse images under g of a collection of small disks around the critical values modulo other points inside the small disks. The desired conclusion follows. □

§2. A Weak Version of the Main Theorem via an Index Theorem

In Theorem 2.4 of this section, we show how the Euler characteristic information given by the vanishing index theorem of Lê [**L1**], Sabbah [**Sa**], and Ginsburg [**G**] (see Theorem 2.1) can actually be used to find the individual Betti numbers; all that is required is a quick application of perverse cohomology. Thus, in this section, we give a short, elegant proof of the main theorem, **if one is willing to ignore torsion**. The other sections of this paper are present precisely because, in general, we consider it unsatisfactory to ignore torsion, and because the index-theoretic approach is not easily adaptable to other questions.

Throughout this section, we assume that our base ring, R, is a principal ideal domain, that X is a complex analytic space embedded in an open subset $\mathcal{U}$ of complex affine space, that $\mathcal{S} := \{S_\alpha\}$ is a complex Whitney stratification of X with connected strata, that $\mathbf{F}^\bullet \in D^b_{\mathcal{S}}(X)$, and that $\hat{f} : \mathcal{U} \to \mathbb{C}$ is complex analytic. Recall that $d_\alpha := \dim S_\alpha$. Let $f := \hat{f}_{|X}$.

As R is a p.i.d., the rank of a finitely-generated R-module is well-defined, and the Euler characteristic – the alternating sum of the ranks – is a well-defined, additive function on complexes of finitely-generated R-modules. Recall that the *characteristic cycle,* $\mathrm{Ch}(\mathbf{F}^\bullet)$*, of* $\mathbf{F}^\bullet$ in $T^*\mathcal{U}$ is the linear combination $\sum_\alpha m_\alpha(\mathbf{F}^\bullet)\left[\overline{T^*_{S_\alpha}\mathcal{U}}\right]$, where the $m_\alpha(\mathbf{F}^\bullet)$ are integers determined by the Euler characteristic, as follows

$$m_\alpha(\mathbf{F}^\bullet) \;:=\; (-1)^{\dim X - d_\alpha}\chi\big(\mathbb{H}^*(\mathbb{N}_\alpha, \mathbb{L}_\alpha; \mathbf{F}^\bullet)\big) \;=\; (-1)^{\dim X - d_\alpha - 1}\chi(\phi_{L_{|_{\mathbb{N}_\alpha}}}\mathbf{F}^\bullet{}_{|_{\mathbb{N}_\alpha}})_{\mathbf{x}}$$

for any point $\mathbf{x}$ in S_α, with normal slice $\mathbb{N}_\alpha$ at $\mathbf{x}$, and any $L : (\mathcal{U}, x) \to (\mathbb{C}, 0)$ such that $d_{\mathbf{x}}L$ is a non-degenerate covector at $\mathbf{x}$ (with respect to our fixed stratification; see [**G-M2**]) and $L_{|_{S_\alpha}}$ has a Morse singularity at $\mathbf{x}$. This cycle is independent of all the choices made (see, for instance, [**K-S3**], Chapter IX).

Let $\mathrm{Ch}(\mathbf{F}^\bullet)$ denote the characteristic cycle of $\mathbf{F}^\bullet$ in $T^*\mathcal{U}$. Then, the result of Lê [**L1**], Sabbah [**S**], and Ginsburg [**G**] is:

Theorem 2.1. *Suppose that* $(\mathbf{p}, d_{\mathbf{p}}\hat{f})$ *is an isolated point in the intersection* $|\mathrm{Ch}(\mathbf{F}^\bullet)| \cap \operatorname{im} d\hat{f}$*, and that* $f(\mathbf{p}) = 0$*. Then, the Euler characteristic of the stalk cohomology of the vanishing cycles of* f *is related to the intersection multiplicity of* $\mathrm{Ch}(\mathbf{F}^\bullet)$ *and image of* $d\hat{f}$ *by*

$$\chi(\phi_f\mathbf{F}^\bullet)_{\mathbf{p}} = (-1)^{\dim X - 1}\left(\mathrm{Ch}(\mathbf{F}^\bullet)\cdot \operatorname{im} d\hat{f}\right)_{(\mathbf{p}, d_{\mathbf{p}}\hat{f})}.$$

Remark 2.2. The proofs of Lê, Sabbah, and Ginsburg all use the language or techniques of $\mathcal{D}$-modules; hence, they all use the complex field for coefficients. Lê's Morse-theoretic proof certainly goes through without change if the base ring is a p.i.d. Moreover, our generalization of 2.1, which appears in [**M1**], Theorem 2.10 also has a purely Morse-theoretic proof and, hence, is clearly true even when the base ring is a p.i.d.

We wish to show how one can use perverse cohomology to extract Betti number information from Theorem 2.1. We list some properties of the perverse cohomology and of vanishing cycles that we will need later. The reader is referred to [**BBD**] and [**K-S3**].

The perverse cohomology functor (using middle perversity, μ) on X, ${}^\mu H^0$, is a functor from $D^b_{\mathbb{C}}(X)$ to the Abelian category of perverse sheaves on X. One lets ${}^\mu H^i(\mathbf{F}^\bullet)$ denote ${}^\mu H^0(\mathbf{F}^\bullet[i])$.

If $\mathbf{F}^\bullet$ is constructible with respect to $\mathcal{S}$, then ${}^\mu H^0(\mathbf{F}^\bullet)$ is also constructible with respect to $\mathcal{S}$, and $\big({}^\mu H^0(\mathbf{F}^\bullet)\big)_{|_{\mathbb{N}_\alpha}}[-d_\alpha]$ is naturally isomorphic to ${}^\mu H^0(\mathbf{F}^\bullet_{|_{\mathbb{N}_\alpha}}[-d_\alpha])$.

The functor ${}^\mu H^0$, applied to a perverse sheaf $\mathbf{P}^\bullet$ is canonically isomorphic to $\mathbf{P}^\bullet$. In addition, a bounded, constructible complex of sheaves $\mathbf{F}^\bullet$ is perverse if and only ${}^\mu H^k(\mathbf{F}^\bullet) = 0$ for all $k \neq 0$. In particular, if X is a local complete intersection, then ${}^\mu H^{\dim X}(\mathbb{Z}^\bullet_X) \cong \mathbb{Z}^\bullet_X[\dim X]$ and ${}^\mu H^k(\mathbb{Z}^\bullet_X) = 0$ if $k \neq \dim X$.

The functor ${}^\mu H^0$ commutes with vanishing cycles with a shift of -1, nearby cycles with a shift of -1, and Verdier dualizing. That is, there are natural isomorphisms

$${}^\mu H^0 \circ \phi_f[-1] \cong \phi_f[-1] \circ {}^\mu H^0, \quad {}^\mu H^0 \circ \psi_f[-1] \cong \psi_f[-1] \circ {}^\mu H^0, \quad \text{and } \mathcal{D} \circ {}^\mu H^0 \cong {}^\mu H^0 \circ \mathcal{D}.$$

Let $\mathbf{F}^\bullet$ be a bounded complex of sheaves on X which is constructible with respect to $\mathcal{S}$. Let $S_{\max}$ be a maximal stratum (i.e., one not contained in the closure of another) which is contained in the support of $\mathbf{F}^\bullet$, and let $m = \dim S_{\max}$. Then, $\big({}^{\mu}\!H^0(\mathbf{F}^\bullet)\big)_{|_{S_{\max}}}$ is isomorphic (in the derived category) to the complex which has $(\mathbf{H}^{-m}(\mathbf{F}^\bullet))_{|_{S_{\max}}}$ in degree $-m$ and zero in all other degrees.

In particular, $\operatorname{supp}\mathbf{F}^\bullet = \bigcup_i \operatorname{supp}{}^{\mu}\!H^i(\mathbf{F}^\bullet)$, and if $\mathbf{F}^\bullet$ is supported on an isolated point, $\mathbf{q}$, then $H^0({}^{\mu}\!H^0(\mathbf{F}^\bullet))_{\mathbf{q}} \cong H^0(\mathbf{F}^\bullet)_{\mathbf{q}}$.

Lemma 2.3. The characteristic cycle of the perverse cohomology of $\mathbf{F}^\bullet[i]$ is given by

$$\operatorname{Ch}({}^{\mu}\!H^i(\mathbf{F}^\bullet)) = (-1)^{\dim X}\sum_{\alpha} b_{i-d_\alpha}(\mathbb{N}_\alpha, \mathbb{L}_\alpha; \mathbf{F}^\bullet)\left[\overline{T^*_{S_\alpha}\mathcal{U}}\right],$$

where $b_j(\mathbb{N}_\alpha, \mathbb{L}_\alpha; \mathbf{F}^\bullet)$ denotes the j-th relative Betti number, i.e., $b_j(\mathbb{N}_\alpha, \mathbb{L}_\alpha; \mathbf{F}^\bullet) = \operatorname{rk}_R \mathbb{H}^j(\mathbb{N}_\alpha, \mathbb{L}_\alpha; \mathbf{F}^\bullet)$.

Proof.

$$m_\alpha({}^{\mu}\!H^0(\mathbf{F}^\bullet[i])) = (-1)^{\dim X - d_\alpha - 1}\chi(\phi_{L_{|_{\mathbb{N}_\alpha}}}{}^{\mu}\!H^0(\mathbf{F}^\bullet[i])_{|_{\mathbb{N}_\alpha}})_{\mathbf{x}} =$$

$$(-1)^{\dim X}\chi\big(\phi_{L_{|_{\mathbb{N}_\alpha}}}[-1]{}^{\mu}\!H^0(\mathbf{F}^\bullet[i])_{|_{\mathbb{N}_\alpha}}[-d_\alpha]\big)_{\mathbf{x}} = (-1)^{\dim X}\chi\big(\phi_{L_{|_{\mathbb{N}_\alpha}}}[-1]{}^{\mu}\!H^0(\mathbf{F}^\bullet_{|_{\mathbb{N}_\alpha}}[i-d_\alpha])_{\mathbf{x}} =$$

$$(-1)^{\dim X}\operatorname{rk} H^0\big(\phi_{L_{|_{\mathbb{N}_\alpha}}}[-1](\mathbf{F}^\bullet_{|_{\mathbb{N}_\alpha}}[i-d_\alpha])\big)_{\mathbf{x}} = (-1)^{\dim X} b_{i-d_\alpha}(\mathbb{N}_\alpha, \mathbb{L}_\alpha; \mathbf{F}^\bullet). \quad \square$$

We can now prove the fundamental result of this section.

Theorem 2.4. *Suppose that $(\mathbf{p}, d_{\mathbf{p}}\hat f)$ is an isolated point in the intersection $|\operatorname{Ch}(\mathbf{F}^\bullet)| \cap \operatorname{im} d\hat f$, and that $f(\mathbf{p}) = 0$. Then, the Betti numbers of the stalk cohomology of the vanishing cycles of f are given by*

$$b_{i-1}(\phi_f\mathbf{F}^\bullet)_{\mathbf{p}} = \sum_\alpha b_{i-d_\alpha}(\mathbb{N}_\alpha, \mathbb{L}_\alpha; \mathbf{F}^\bullet)\big(\overline{T^*_{S_\alpha}\mathcal{U}}\ \cdot\ \operatorname{im} d\hat f\big)_{(\mathbf{p}, d_{\mathbf{p}}\hat f)} =$$

$$\sum_\alpha b_{i-d_\alpha}(\mathbb{N}_\alpha, \mathbb{L}_\alpha; \mathbf{F}^\bullet)\left[\big(\Gamma^1_{f,L}(S_\alpha)\cdot V(f)\big)_{\mathbf{p}} - \big(\Gamma^1_{f,L}(S_\alpha)\cdot V(L)\big)_{\mathbf{p}}\right].$$

Proof. We use the properties of perverse cohomology:

$$b_{i-1}(\phi_f\mathbf{F}^\bullet)_{\mathbf{p}} = \operatorname{rk} H^0(\phi_f\mathbf{F}^\bullet[i-1])_{\mathbf{p}} = \operatorname{rk} H^0\big({}^{\mu}\!H^0(\phi_f\mathbf{F}^\bullet[i-1])\big)_{\mathbf{p}} =$$

$$\chi\big({}^{\mu}\!H^0(\phi_f\mathbf{F}^\bullet[i-1])\big)_{\mathbf{p}} = \chi\big(\phi_f[-1]{}^{\mu}\!H^0(\mathbf{F}^\bullet[i])\big)_{\mathbf{p}} =$$

$$(-1)^{\dim X}\big(\operatorname{Ch}({}^{\mu}\!H^0(\mathbf{F}^\bullet[i]))\cdot\operatorname{im} d\hat f\big)_{(\mathbf{p}, d_{\mathbf{p}}\hat f)},$$

where the last equality follows from Theorem 2.1. Now, the result follows from the lemma. $\square$

Remark 2.5. The reader should note that by taking the base ring to be the finite field $\mathbb{Z}/p$, one can use Theorem 2.4 to detect p-torsion in the integral cohomology of the Milnor fibre (see section 3).

Unfortunately, one cannot quite recover the integral cohomology structure this way; while $\mathbb{Z}/p$ coefficients would tell one how many direct summands of the form $\mathbb{Z}/p^k$ appear in the integral cohomology, they would not distinguish between the various powers of p.

It may appear that we have not proved what we claimed we would prove; namely, that we would conclude Theorem 2.4 under the hypothesis that $\mathbf{p}$ was an isolated point in the support $\phi_f\mathbf{F}^\bullet$. However, we have the following:

Proposition 2.6. *If R is a p.i.d., then*

$$\big\{\mathbf{p}\in X \mid f(\mathbf{p})=0, (\mathbf{p}, d_{\mathbf{p}}\hat{f})\in|\operatorname{Ch}(\mathbf{F}^\bullet)|\big\} \ \subseteq\ \operatorname{supp}\phi_f\mathbf{F}^\bullet.$$

If R is a field and $\mathbf{P}^\bullet$ is a perverse sheaf on X, then

$$\big\{\mathbf{p}\in X \mid f(\mathbf{p})=0, (\mathbf{p}, d_{\mathbf{p}}\hat{f})\in|\operatorname{Ch}(\mathbf{P}^\bullet)|\big\} \ =\ \operatorname{supp}\phi_f\mathbf{P}^\bullet.$$

Proof. The first statement is immediate from Theorem 2.10 of [**M1**]. The second statement follows trivially from Theorem 3.2 of [**M1**]. □

§3. Invisible Strata, Field Coefficients, and the Vanishing Cycles

We continue with the notation given at the beginning of Section 2, including that R is a p.i.d.

In this section, we begin by discussing how changing the base ring, R, affects the vanishing cycles, characteristic cycles, and the perverse cohomology; this discussion is necessary if we wish to obtain a result which includes possible torsion in the cohomology. This discussion leads us to define, in 3.3, $\mathbf{F}^\bullet$-visible strata as those with non-trivial $\mathbf{F}^\bullet$-Morse modules. We can then prove the main result of this section, Theorem 3.4, which describes the support of the vanishing cycles; precisely, the result of 3.4 is

$$\bigcup_{v\in\mathbb{C}}\operatorname{supp}\phi_{f-v}\mathbf{F}^\bullet \ =\ \Big\{\mathbf{x}\in X \mid (x, d_{\mathbf{x}}\hat{f})\in \bigcup_{\substack{\mathbf{F}^\bullet\text{-}visible\\ S_\alpha}} \overline{T^*_{S_\alpha}\mathcal{U}}\Big\}.$$

Basic Results

For each prime ideal $\mathfrak{p}$ of R, let $k_{\mathfrak{p}}$ denote the field of fractions of $R/\mathfrak{p}$, i.e., k_0 is the field of fractions of R, and for $\mathfrak{p}\neq 0$, $k_{\mathfrak{p}}=R/\mathfrak{p}$. There are the obvious functors $\delta_{\mathfrak{p}}:\mathbf{D}^b_{\mathbb{C}}(R_X)\to\mathbf{D}^b_{\mathbb{C}}((k_{\mathfrak{p}})_X)$, which sends $\mathbf{F}^\bullet$ to $\mathbf{F}^\bullet\overset{L}{\otimes}(k_{\mathfrak{p}})^\bullet_X$, and $\epsilon_{\mathfrak{p}}:\mathbf{D}^b_{\mathbb{C}}((k_{\mathfrak{p}})_X)\to\mathbf{D}^b_{\mathbb{C}}(R_X)$, which considers $k_{\mathfrak{p}}$-vector spaces as R-modules.

If $\mathbf{A}^\bullet$ is a complex of $k_{\mathfrak{p}}$-vector spaces, we may consider the perverse cohomology of $\mathbf{A}^\bullet$, ${}^\mu H^i_{k_{\mathfrak{p}}}(\mathbf{A}^\bullet)$, or the perverse cohomology of $\epsilon(\mathbf{A}^\bullet)$, which we denote by ${}^\mu H^i_R(\mathbf{A}^\bullet)$. If $\mathbf{A}^\bullet\in\mathbf{D}^b_{\mathbb{C}}((k_{\mathfrak{p}})_X)$ and $S_{\max}$ is a maximal stratum contained in the support of $\mathbf{A}^\bullet$, then there is a canonical isomorphism

$$\epsilon\big(({}^\mu H^i_{k_{\mathfrak{p}}}(\mathbf{A}^\bullet))_{|S_\alpha}\big)\cong({}^\mu H^i_R(\mathbf{A}^\bullet))_{|S_\alpha};$$

in particular, $\operatorname{supp}{}^{\mu}H^i_{k_{\mathfrak p}}(\mathbf{A}^\bullet)=\operatorname{supp}{}^{\mu}H^i_R(\mathbf{A}^\bullet)$.

If $\mathbf{F}^\bullet\in\mathbf{D}^b_{\mathbb C}(R_X)$, $S_{\max}$ is a maximal stratum contained in the support of $\mathbf{F}^\bullet$, and $\mathbf{x}\in S_{\max}$, then for some prime ideal $\mathfrak p\subset R$ and for some integer i, $H^i(\mathbf{F}^\bullet)_{\mathbf x}\otimes k_{\mathfrak p}\neq 0$; it follows that $S_{\max}$ is also a maximal stratum in the support of $\mathbf{F}^\bullet\overset{L}{\otimes}(k_{\mathfrak p})^\bullet_X$. Thus,

$$\operatorname{supp}\mathbf{F}^\bullet=\bigcup_{\mathfrak p}\operatorname{supp}(\mathbf{F}^\bullet\overset{L}{\otimes}(k_{\mathfrak p})^\bullet_X)$$

and so

$$\operatorname{supp}\mathbf{F}^\bullet=\bigcup_{i,\mathfrak p}\operatorname{supp}{}^{\mu}H^i_{k_{\mathfrak p}}(\mathbf{F}^\bullet\overset{L}{\otimes}(k_{\mathfrak p})^\bullet_X),$$

where the boundedness and constructibility of $\mathbf{F}^\bullet$ imply that this union is locally finite.

For each prime ideal $\mathfrak p$, there is a natural isomorphism in $\mathbf{D}^b_{\mathbb C}(R_{V(f)})$ given by

$$\phi_f\big(\mathbf{F}^\bullet\overset{L}{\otimes}(k_{\mathfrak p})^\bullet_X\big)\cong\big(\phi_f\mathbf{F}^\bullet\big)\overset{L}{\otimes}(k_{\mathfrak p})^\bullet_{V(f)}$$

(this is a particularly trivial case of the Sebastiani-Thom Isomorphism of [**M3**]), and hence, the stalk cohomology is given by

$$H^i\big(\phi_f\big(\mathbf{F}^\bullet\overset{L}{\otimes}(k_{\mathfrak p})^\bullet_X\big)\big)_{\mathbf x}\cong\big(H^i(\phi_f\mathbf{F}^\bullet)_{\mathbf x}\otimes k_{\mathfrak p}\big)\ \oplus\ \operatorname{Tor}\big(H^{i+1}(\phi_f\mathbf{F}^\bullet)_{\mathbf x},\ k_{\mathfrak p}\big).$$

Proposition 3.1. *For all integers i and for all prime ideals $\mathfrak p$ in R, the characteristic cycle of the perverse cohomology of the sheaf of $k_{\mathfrak p}$-vector spaces $\mathbf{F}^\bullet[i]\overset{L}{\otimes}(k_{\mathfrak p})^\bullet_X$ is given by*

$$\operatorname{Ch}({}^{\mu}H^i_{k_{\mathfrak p}}(\mathbf{F}^\bullet\overset{L}{\otimes}(k_{\mathfrak p})^\bullet_X))=(-1)^{\dim X}\sum_\alpha m_{i-d_\alpha}\left[\overline{T^*_{S_\alpha}\mathcal U}\right],$$

where $m_{i-d_\alpha}:=\dim_{k_{\mathfrak p}}(H^{i-d_\alpha}(\mathbb N_\alpha,\mathbb L_\alpha;\mathbf{F}^\bullet)\otimes k_{\mathfrak p})+\dim_{k_{\mathfrak p}}\operatorname{Tor}(H^{i-d_\alpha+1}(\mathbb N_\alpha,\mathbb L_\alpha;\mathbf{F}^\bullet),k_{\mathfrak p})$.

Proof. Given the basic facts at the beginning of the section, the proof is exactly that of Lemma 2.3. □

Proposition 3.2 (Continuity of Vanishing Support). *Suppose that $\mathbf{x}_j\in X$ and $\mathbf{x}_j\to\mathbf{x}$. For each j, let $\hat f_j$ be a complex analytic function locally defined on $\mathcal U$ at $\mathbf{x}_j$, such that $\hat f_j(\mathbf{x}_j)=0$, and let $f_j:=\hat f_{j_{|X}}$. Suppose that $\mathbf{x}_j\in\operatorname{supp}\phi_{f_j}\mathbf{F}^\bullet$ and that $d_{\mathbf{x}_j}\hat f_j\to d_{\mathbf x}\hat f$.*

Then, the following three equivalent conclusions hold:

a) $\mathbf{x}\in\operatorname{supp}\phi_{f-f(\mathbf x)}\mathbf{F}^\bullet$;

b) *there exists an integer i and the prime ideal $\mathfrak p$ in R such that*

$$\mathbf{x}\in\operatorname{supp}\phi_{f-f(\mathbf x)}{}^{\mu}H^i_{k_{\mathfrak p}}(\mathbf{F}^\bullet\overset{L}{\otimes}(k_{\mathfrak p})^\bullet_X);$$

and

c) *there exists an integer* i *and the prime ideal* $\mathfrak{p}$ *in* R *such that*

$$(\mathbf{x}, d_{\mathbf{x}}\hat{f}) \in \big|\operatorname{Ch}({}^{\mu}H^i_{k_{\mathfrak{p}}}(\mathbf{F}^\bullet \overset{L}{\otimes} (k_{\mathfrak{p}})^\bullet_X))\big|.$$

Proof. Let us prove the equivalence of a), b), and c) first. By 2.6, b) and c) are equivalent. Now, a) and b) are equivalent because

$$\operatorname{supp}\phi_{f-f(\mathbf{x})}\mathbf{F}^\bullet = \bigcup_{i,\mathfrak{p}}\operatorname{supp}{}^{\mu}H^i_{k_{\mathfrak{p}}}(\phi_{f-f(\mathbf{x})}\mathbf{F}^\bullet \overset{L}{\otimes} (k_{\mathfrak{p}})^\bullet_{V(f-f(\mathbf{x}))}) =$$

$$\bigcup_{i,\mathfrak{p}}\operatorname{supp}{}^{\mu}H^i_{k_{\mathfrak{p}}}\left(\phi_{f-f(\mathbf{x})}(\mathbf{F}^\bullet \overset{L}{\otimes} (k_{\mathfrak{p}})^\bullet_X)\right) = \bigcup_{i,\mathfrak{p}}\operatorname{supp}\phi_{f-f(\mathbf{x})}{}^{\mu}H^i_{k_{\mathfrak{p}}}\left(\mathbf{F}^\bullet \overset{L}{\otimes} (k_{\mathfrak{p}})^\bullet_X\right).$$

We will now prove that the assumptions of the first paragraph imply c). As above, we have

$$\operatorname{supp}\phi_{f_j}\mathbf{F}^\bullet = \bigcup_{i,\mathfrak{p}}\operatorname{supp}\phi_{f_j}{}^{\mu}H^i_{k_{\mathfrak{p}}}\left(\mathbf{F}^\bullet \overset{L}{\otimes} (k_{\mathfrak{p}})^\bullet_X\right).$$

Now, near $\mathbf{x}$, $\operatorname{supp}\mathbf{F}^\bullet$ is a finite union $\bigcup_{i,\mathfrak{p}}\operatorname{supp}{}^{\mu}H^i_{k_{\mathfrak{p}}}(\mathbf{F}^\bullet \overset{L}{\otimes} (k_{\mathfrak{p}})^\bullet_X)$; thus, by taking a subsequence, we may assume that there is one i and one $\mathfrak{p}$ such that, for all j, $\mathbf{x}_j \in \operatorname{supp}\phi_{f_j}{}^{\mu}H^i_{k_{\mathfrak{p}}}\left(\mathbf{F}^\bullet \overset{L}{\otimes} (k_{\mathfrak{p}})^\bullet_X\right)$. It follows from 2.6 that $(\mathbf{x}_j, d_{\mathbf{x}_j}\hat{f}_j) \in \big|\operatorname{Ch}\left({}^{\mu}H^i_{k_{\mathfrak{p}}}\left(\mathbf{F}^\bullet \overset{L}{\otimes} (k_{\mathfrak{p}})^\bullet_X\right)\right)\big|$. As $\big|\operatorname{Ch}\left({}^{\mu}H^i_{k_{\mathfrak{p}}}\left(\mathbf{F}^\bullet \overset{L}{\otimes} (k_{\mathfrak{p}})^\bullet_X\right)\right)\big|$ is closed, we conclude that $(\mathbf{x}, d_{\mathbf{x}}\hat{f}) \in \big|\operatorname{Ch}\left({}^{\mu}H^i_{k_{\mathfrak{p}}}\left(\mathbf{F}^\bullet \overset{L}{\otimes} (k_{\mathfrak{p}})^\bullet_X\right)\right)\big|$. □

Looking at Theorem 2.4, and Propositions 3.1 and 3.2, we see that any stratum S_α for which $H^*(\mathbb{N}_\alpha, \mathbb{L}_\alpha; \mathbf{F}^\bullet) = 0$ is essentially irrelevant as far as vanishing cycles are concerned. Hence, we make the following definition.

Definition 3.3. A stratum S_α is $\mathbf{F}^\bullet$-*invisible* if $H^*(\mathbb{N}_\alpha, \mathbb{L}_\alpha; \mathbf{F}^\bullet) = 0$. Otherwise, we say that S_α is $\mathbf{F}^\bullet$-*visible*.

The basic principle is that $\mathbf{F}^\bullet$-invisible strata do not contribute any cohomology in Morse Theory arguments. As an example, we have

Theorem 3.4.
$$\bigcup_{v\in\mathbb{C}}\operatorname{supp}\phi_{f-v}\mathbf{F}^\bullet = \left\{\mathbf{x}\in X \mid (x, d_{\mathbf{x}}\hat{f}) \in \bigcup_{\substack{\mathbf{F}^\bullet\text{-}visible\\ S_\alpha}}\overline{T^*_{S_\alpha}\mathcal{U}}\right\}.$$

Proof. This is immediate from the equivalence of a) and c) in 3.2, and the description of the characteristic cycle given in 3.1. □

Remark 3.5. The union on the left side above is not just locally finite, but, in fact, locally consists of a single support, i.e., near a point $\mathbf{p} \in X$, $\operatorname{supp} \phi_{f-v}\mathbf{F}^\bullet = \emptyset$ unless $v = f(\mathbf{p})$.

We should also point out an elementary, but important, relation with the real structure of the conormal spaces. In the situation of Theorem 3.4, $d_{\mathbf{x}}\hat{f} \in \big(\overline{T^*_{S_\alpha}\mathcal{U}}\big)_{\mathbf{x}}$ if and only if $d_{\mathbf{x}}(\operatorname{Re}\hat{f}) \in \big(\overline{T^*_{S_\alpha}\mathcal{U}}\big)_{\mathbf{x}}$ (considered with its real structure); this follows from the Cauchy-Riemann equations. Of course, $d_{\mathbf{x}}\hat{f} \in \big(\overline{T^*_{S_\alpha}\mathcal{U}}\big)_{\mathbf{x}}$ if and only if there exists $a+bi \neq 0$ such that $d_{\mathbf{x}}[(a+bi)\hat{f}] = (a+bi)d_{\mathbf{x}}\hat{f} \in \big(\overline{T^*_{S_\alpha}\mathcal{U}}\big)_{\mathbf{x}}$. It follows that $d_{\mathbf{x}}\hat{f} \notin \big(\overline{T^*_{S_\alpha}\mathcal{U}}\big)_{\mathbf{x}}$ if and only if, for all $(a,b) \in \mathbb{R}^2 - \{\mathbf{0}\}$, $a\, d_{\mathbf{x}}(\operatorname{Re}\hat{f}) + b\, d_{\mathbf{x}}(\operatorname{Im}\hat{f}) \notin \big(\overline{T^*_{S_\alpha}\mathcal{U}}\big)_{\mathbf{x}}$. This will be importance to us in section 5.

§4. **Microlocal Basics**

In this section, we will discuss conormal geometry, and provide a down-to-Earth discussion of a number of microlocal results of Kashiwara and Schapira. We continue with our earlier notations, including that the base ring R is a p.i.d.

We will always work inside an open subset $\mathcal{U}$ in $\mathbb{R}^{m+1}$. Let $T^*\mathcal{U} \xrightarrow{\pi} \mathcal{U}$ denote the conormal bundle. This is isomorphic to $\mathcal{U} \times \mathbb{R}^{m+1} \xrightarrow{\mathrm{pr}} \mathcal{U}$; we use $x_0, \dots, x_m, \xi_0, \dots, \xi_m$ for coordinates on $T^*\mathcal{U}$. We use $\mathbb{RP}(T^*\mathcal{U})$ to denote the total space of the projectivized cotangent bundle; hence, $\mathbb{RP}(T^*\mathcal{U}) \cong \mathcal{U} \times \mathbb{RP}^m$.

If M is a smooth submanifold of $\mathcal{U}$, then the *conormal bundle to M in $\mathcal{U}$* is given by

$$T^*_M\mathcal{U} := \{(\mathbf{p}, \eta) \in T^*\mathcal{U} \mid \mathbf{p} \in M,\ \eta(T_{\mathbf{p}}M) = 0\}.$$

Using this notation, the zero-section is given by $T^*_{\mathcal{U}}\mathcal{U}$. In addition, two submanifolds $M, N \subseteq \mathcal{U}$ intersect transversely if and only if $T^*_M\mathcal{U} \cap T^*_N\mathcal{U} \subseteq T^*_{\mathcal{U}}\mathcal{U}$.

There is a canonical 1-form, α, on $T^*\mathcal{U}$, which we will describe at each point. Let $(\mathbf{p}, \eta) \in T^*\mathcal{U}$. We want to describe the linear function $\alpha_{(\mathbf{p},\eta)} : T_{(\mathbf{p},\eta)}(T^*\mathcal{U}) \to \mathbb{R}$. Note that $\eta : T_{\mathbf{p}}\mathcal{U} \to \mathbb{R}$ and that $d_{(\mathbf{p},\eta)}\pi : T_{(\mathbf{p},\eta)}(T^*\mathcal{U}) \to T_{\mathbf{p}}\mathcal{U}$. Thus, we may define $\alpha_{(\mathbf{p},\eta)} := \eta \circ d_{(\mathbf{p},\eta)}\pi$. One easily verifies that, in coordinates, α is given by $\xi_0 dx_0 + \cdots + \xi_m dx_m$.

Let $\mathbb{R}^+$ denote the strictly positive real numbers. A subset S of $T^*\mathcal{U}$ is $\mathbb{R}^+$*-conic* provided that, for all $\mathbf{p} \in \mathcal{U}$, for all $r \in \mathbb{R}^+$ and $\eta \in \pi^{-1}(\mathbf{p})$, $r\eta \in S$, i.e., the fibres of S are closed under positive scalar multiplication. While the zero vector need not be in a fibre if a conic subset, if S is closed and conic, then the zero vector is in every non-empty fibre. In addition, it is easy to prove that the image under π of a closed, conic subset of $T^*\mathcal{U}$ is a closed subset of $\mathcal{U}$. If S is $\mathbb{R}^+$-conic, then we may consider its projectivization $\mathbb{RP}(S) \subseteq \mathbb{RP}(T^*\mathcal{U})$.

We need to define and describe isotropic subsets of $T^*\mathcal{U}$.

Definition 4.1. A subset $S \subseteq T^*\mathcal{U}$ is *isotropic* if $\alpha_{|_S} = 0$.

The following proposition provides a nice characterization of isotropic sets; it is 8.3.10 of [**K-S3**].

Proposition 4.2. *Let S be an $\mathbb{R}^+$-conic subanalytic subset of $T^*\mathcal{U}$. Then, the following are equivalent:*

i) S is isotropic;

ii) there exists a locally finite family $\{W^j\}$ of subanalytic subsets of $\mathcal{U}$ such that $S \subseteq \bigcup_j \overline{T^*_{W^j_{\mathrm{reg}}}\mathcal{U}}$;

iii) there exists a finite family $\{W^j\}$ *of subanalytic submanifolds of* $\mathcal{U}$ *such that* $W^j \subseteq \pi(S)$ *and* $S \subseteq \bigcup_j T^*_{W^j}\mathcal{U}$.

Our primary interest in isotropic subsets stems from the following theorem ([**K-S3**], 8.3.12), which essentially says that if S is a closed, $\mathbb{R}^+$-conic, subanalytic, isotropic subset of $T^*\mathcal{U}$, then the critical values of any proper real analytic function, relative to S, are discrete.

Theorem 4.3 (Microlocal Bertini-Sard Theorem). *Let* $\phi : \mathcal{U} \to \mathbb{R}$ *be a real analytic function and let* $S \subseteq T^*\mathcal{U}$ *be a closed,* $\mathbb{R}^+$*-conic, subanalytic, isotropic subset. Assume that* ϕ *is proper on* $\pi(S)$.

Then, the set $\{t \in \mathbb{R} \mid \text{there exists } \mathbf{x} \in \mathcal{U} \text{ such that } t = \phi(\mathbf{x}) \text{ and } (\mathbf{x}, d_{\mathbf{x}}\phi) \in S\}$ *is discrete.*

If $E \subseteq T^*(\mathcal{U})$, then E^a denotes the image of E by the antipodal map, i.e.,

$$E^a := \{(\mathbf{p}, -\eta) \mid (\mathbf{p}, \eta) \in E\}.$$

If A and B are $\mathbb{R}^+$-conic subsets of $T^*\mathcal{U}$, then $A + B$ is the $\mathbb{R}^+$-conic subset of $T^*\mathcal{U}$ which lies over $\pi(A) \cap \pi(B)$ and such that

$$\pi^{-1}(\mathbf{x}) \cap (A + B) = \{(\mathbf{x}, a + b) \mid (\mathbf{x}, a) \in A,\ (\mathbf{x}, b) \in B\}.$$

If A and B are closed, then certainly the fibres of $A + B$ are closed; however, $A + B$, itself, need not be closed (but see 4.6 below).

We need to describe the $\widehat{+}$ operation of [**K-S3**], 6.2.3.v. While the definition of $\widehat{+}$ is somewhat difficult to unravel, the characterization of the operation given in 6.2.8.ii of [**K-S3**] is easy to understand; hence, we will use this as our definition.

Definition 4.4. Let A and B be two closed $\mathbb{R}^+$-conic subsets of $T^*\mathcal{U}$. Then, $A \widehat{+} B$ is the subset of $T^*\mathcal{U}$ defined by: $(\mathbf{p}, \eta) \in A \widehat{+} B$ if and only if there exist sequences $(\mathbf{x}_n, \sigma_n) \in A$ and $(\mathbf{y}_n, \tau_n) \in B$ such that $\mathbf{x}_n \to \mathbf{p}$, $\mathbf{y}_n \to \mathbf{p}$, $\sigma_n + \tau_n \to \eta$, and $|\mathbf{x}_n - \mathbf{y}_n| \cdot |\sigma_n| \to 0$.

Remark 4.5. It follows from the definition that $A \widehat{+} B$ is closed, and lies over the closed set $\pi(A) \cap \pi(B)$. Also, while the definition may appear to be asymmetric, in fact, the conditions imply that $|\mathbf{x}_n - \mathbf{y}_n| \cdot |\tau_n| \to 0$. Note that $A + B \subseteq A \widehat{+} B$.

The following is a combination of Lemma 5.4.7 and Corollary 8.3.18.i of [**K-S3**].

Proposition 4.6. Let A and B be two closed, $\mathbb{R}^+$-conic subsets of $T^*\mathcal{U}$.

i) Then, $A \widehat{+} B$ is also closed and $\mathbb{R}^+$-conic.

ii) If $A \cap B^a \subseteq T^*_{\mathcal{U}}\mathcal{U}$, then $A + B$ is closed and $\mathbb{R}^+$-conic.

iii) If A and B are subanalytic, isotropic subsets of $T^*\mathcal{U}$, then $A\widehat{+}B$ is a closed, $\mathbb{R}^+$-conic, subanalytic, isotropic subset of $T^*\mathcal{U}$.

For $\mathbf{F}^\bullet \in D^b(\mathcal{U})$, we need to define the micro-support, $SS(\mathbf{F}^\bullet)$, of $\mathbf{F}^\bullet$; the micro-support consists of a set of covectors which point in directions in which $\mathbf{F}^\bullet$ "does not propagate"– that is, directions in which the hypercohomology of $\mathbf{F}^\bullet$ changes locally.

Definition 4.7. The *micro-support of* $\mathbf{F}^\bullet$, $SS(\mathbf{F}^\bullet)$, is the subset of $T^*\mathcal{U}$ defined by the following: $(\mathbf{p},\eta) \notin SS(\mathbf{F}^\bullet)$ if and only if there exists an open neighborhood Ω of $(\mathbf{p},\eta)$ in $T^*\mathcal{U}$ such that for all $\mathbf{x} \in \mathcal{U}$, for all real C^1 functions ψ defined in a neighborhood of $\mathbf{x}$ such that $(\mathbf{x}, d_{\mathbf{x}}\psi) \in \Omega$ and $\psi(\mathbf{x}) = 0$,

$$\big(R\Gamma_{\{\mathbf{y}\in\mathcal{U}\,|\,\psi(\mathbf{y})\geqslant 0\}}(\mathbf{F}^\bullet)\big)_{\mathbf{x}} = 0.$$

The following proposition is 5.1.3.i of [**K-S3**], combined with part of 8.4.2.

Proposition 4.8. *Let* $\mathbf{F}^\bullet \in D^b(\mathcal{U})$. *Then,* $SS(\mathbf{F}^\bullet)$ *is a closed,* $\mathbb{R}^+$*-conic subset of* $T^*\mathcal{U}$ *and* $SS(\mathbf{F}^\bullet) \cap T^*_{\mathcal{U}}\mathcal{U} = \operatorname{supp}\mathbf{F}^\bullet \times \{0\}$.

In addition, if $\mathbf{F}^\bullet$ *is* $\mathbb{R}$*-constructible, then* $SS(\mathbf{F}^\bullet)$ *is a subanalytic, isotropic subset in* $T^*\mathcal{U}$.

The following lemma is 5.4.19 of [**K-S3**]. It tells us that if a function produces no infinitesimal changes in $\mathbf{F}^\bullet$, then it produces no global changes in $\mathbf{F}^\bullet$.

Lemma 4.9. *Let* $\mathbf{F}^\bullet \in D^b(\mathcal{U})$ *and let* $\phi : \mathcal{U} \to \mathbb{R}$ *be a* C^1 *function such that* $\phi_{|_{\operatorname{supp}(\mathbf{F}^\bullet)}}$ *is proper. Let* $a, b \in \mathbb{R}$ *with* $a < b$.

If, for all $\mathbf{x} \in \phi^{-1}([a,b))$, $(\mathbf{x}, d_{\mathbf{x}}\phi) \notin SS(\mathbf{F}^\bullet)$, *then the natural morphisms*

$$R\Gamma\big(\phi^{-1}((-\infty,b);\mathbf{F}^\bullet)\big) \to R\Gamma\big(\phi^{-1}((-\infty,a];\mathbf{F}^\bullet)\big) \to R\Gamma\big(\phi^{-1}((-\infty,a);\mathbf{F}^\bullet)\big)$$

are isomorphisms.

The following is part of Lemma 8.4.7 of [**K-S3**].

Lemma 4.10. *Let* X *be a real analytic space,* $\mathbf{F}^\bullet \in D^b_{\mathbb{R}}(X)$, *and* $r : X \to \mathbb{R}^n$ *a real analytic function. Assume that* $r_{|_{\operatorname{supp}\mathbf{F}^\bullet}}$ *is proper. Then, for all sufficiently small* $\epsilon > 0$, *the inclusion maps induce natural isomorphisms*

$$R\Gamma(r^{-1}(B_\epsilon);\ \mathbf{F}^\bullet) \ \cong\ R\Gamma(r^{-1}(\overset{\circ}{B}_\epsilon);\ \mathbf{F}^\bullet) \ \cong\ R\Gamma(r^{-1}(\mathbf{0});\ \mathbf{F}^\bullet).$$

Our application of 4.9 will be to the case where we begin with a complex $\mathbf{F}^\bullet \in D^b_{\mathbb{C}}(\mathcal{U})$, then restrict $\mathbf{F}^\bullet$ to a closed ball, and then push it forward, back onto $\mathcal{U}$. Thus, we need to know how restricting to a closed ball, and then pushing forward, affects the micro-support. For this, we need two or three more definitions.

Definition 4.11. Suppose that $Z \subseteq \mathcal{U}$ is a closed submanifold with boundary, of dimension equal to that of $\mathcal{U}$. For each point $\mathbf{x} \in \partial Z$, there is a unique, inwardly-pointing unit vector, $\mathbf{n_x}$, which is normal to ∂Z at $\mathbf{x}$. The standard inner-product (dot product) gives a well-defined linear form $\eta_{\mathbf{x}}(\mathbf{v}) := \mathbf{v} \cdot \mathbf{n_x}$.

The subset $N^*(Z) \subseteq T^*(\mathcal{U})$ is defined by $\pi(N^*(Z)) = \mathcal{U}$ and, for all $\mathbf{x} \in \mathcal{U}$,

$$\pi^{-1}(\mathbf{x}) \cap N^*(Z) = \begin{cases} \{(\mathbf{x}, 0)\}, & \text{if } \mathbf{x} \notin \partial Z \\ \{(\mathbf{x}, r\eta_{\mathbf{x}}) \mid r \in \{0\} \cup \mathbb{R}^+\}, & \text{if } \mathbf{x} \in \partial Z. \end{cases}$$

(See 5.3.6 of [**K-S3**].) Hence, $N^*(Z)$ consists of the zero-section of $T^*\mathcal{U}$, together with the "inwardly-pointing conormals" along the boundary of Z. Note that $N^*(Z)$ is closed and $\mathbb{R}^+$-conic. In addition, if Z and ∂Z are subanalytic, then Proposition 4.2 implies that $N^*(Z)$ is isotropic.

Recall that if $j : Z \hookrightarrow \mathcal{U}$ is a closed inclusion, then $\mathbf{F}^\bullet_Z \cong j_!j^*\mathbf{F}^\bullet \cong j_*j^*\mathbf{F}^\bullet$; therefore, Proposition 5.4.8.b.ii of [**K-S3**] yields the following:

Proposition 4.12. *Let $\mathbf{F}^\bullet \in D^b(\mathcal{U})$. Let $j : Z \hookrightarrow \mathcal{U}$ be a closed submanifold with boundary, of dimension equal to that of $\mathcal{U}$.*

If $SS(\mathbf{F}^\bullet) \cap N^*(Z)^a \subseteq T^*_{\mathcal{U}}\mathcal{U}$, then $SS(j_*j^*\mathbf{F}^\bullet) \subseteq N^*(Z) + SS(\mathbf{F}^\bullet)$.

One should think of the condition that $SS(\mathbf{F}^\bullet) \cap N^*(Z)^a \subseteq T^*_{\mathcal{U}}\mathcal{U}$ as a transversality condition between ∂Z and some "virtual strata" whose conormals form $SS(\mathbf{F}^\bullet)$.

In the constructible complex analytic setting, we are able to link the micro-support with the visible strata of Definition 3.3.

Theorem 4.13. *Suppose that $\mathbf{F}^\bullet \in D^b_{\mathbb{C}}(\mathcal{U})$ and that $\{S_\alpha\}$ is a complex analytic Whitney stratification of $\mathcal{U}$ with connected strata. Then,*

$$SS(\mathbf{F}^\bullet) = \bigcup_{\substack{\mathbf{F}^\bullet\text{-}visible \\ S_\alpha}} \overline{T^*_{S_\alpha}\mathcal{U}}.$$

Proof. Let $(\mathbf{x}, \eta) \in T^*\mathcal{U}$. By Proposition 3.1,

$$(\dagger) \qquad \bigcup_{\substack{\mathbf{F}^\bullet\text{-}visible \\ S_\alpha}} \overline{T^*_{S_\alpha}\mathcal{U}} = \bigcup_{i, \mathfrak{p}} \left| \operatorname{Ch}({}^{\mu}H^i_{k_{\mathfrak{p}}}(\mathbf{F}^\bullet \overset{L}{\otimes} (k_{\mathfrak{p}})^\bullet_X)) \right|,$$

where the union is over all integers i and the prime ideals $\mathfrak{p}$ in R.

By 8.6.4 of [**K-S3**], $(\mathbf{x}, \eta) \in SS(\mathbf{F}^\bullet)$ if and only if there exist $\mathbf{x}_j \in \mathcal{U}$ and locally defined complex analytic $\hat{f}_j$ such that $f_j(\mathbf{x}_j) = 0$, $H^*(\phi_{f_j}\mathbf{F}^\bullet)_{\mathbf{x}_j} \neq 0$, and $d_{\mathbf{x}_j}\hat{f}_j \to \eta$; clearly, this is equivalent to: there exist $\mathbf{x}_j \in \mathcal{U}$ and locally defined complex analytic $\hat{f}_j$ such that $f_j(\mathbf{x}_j) = 0$, $\mathbf{x}_j \in \operatorname{supp} \phi_{f_j}\mathbf{F}^\bullet$, and

$d_{\mathbf{x}_j}\hat{f}_j \to \eta$. By the Continuity of Vanishing Support (3.2), this implies that there exists an integer i and the prime ideal $\mathfrak{p}$ in R such that

$$(\mathbf{x}, \eta) \in \Big|\operatorname{Ch}({}^{\mu}H^i_{k_{\mathfrak{p}}}(\mathbf{F}^\bullet \overset{L}{\otimes} (k_{\mathfrak{p}})^\bullet_X))\Big|.$$

Hence, $SS(\mathbf{F}^\bullet) \subseteq \bigcup_{\substack{\mathbf{F}^\bullet\text{-}visible \\ S_\alpha}} \overline{T^*_{S_\alpha}\mathcal{U}}$.

Conversely, if $(\mathbf{x}, \eta) \in \bigcup_{\substack{\mathbf{F}^\bullet\text{-}visible \\ S_\alpha}} \overline{T^*_{S_\alpha}\mathcal{U}}$, then, by (†) and 3.2, $\mathbf{x} \in \operatorname{supp}\phi_{L-L(\mathbf{x})}\mathbf{F}^\bullet$ for any linear form L such that $d_{\mathbf{x}}L = \eta$. Now, 8.6.4 of [**K-S3**] immediately implies that $(\mathbf{x}, \eta) \in SS(\mathbf{F}^\bullet)$. □

Remark 4.14. Theorem 4.13 immediately implies a stronger version of itself. One does not need to begin with a Whitney stratification, but merely any stratification for which the normal data of strata with respect to $\mathbf{F}^\bullet$ is "well-defined", i.e., stratifications in which the normal data in normal slices to strata locally trivializes along the strata. Such stratifications only require refinement by including $\mathbf{F}^\bullet$-invisible strata in order to obtain a Whitney stratification.

This is essentially what is required by Briançon, Maisonobe, and Merle in [**BMM**], where they use stratifications which satisfy Whitney's condition a) and the *property of local stratified triviality.*

Corollary 4.15. *In the notation and situation of Theorem 3.4,*

$$\bigcup_{v\in\mathbb{C}} \operatorname{supp}\phi_{f-v}\mathbf{F}^\bullet \;=\; \Big\{\mathbf{x} \in X \mid (\mathbf{x}, d_{\mathbf{x}}\hat{f}) \in SS(\mathbf{F}^\bullet)\Big\}.$$

Proof. This follows immediately from 3.4 and 4.13, once we extend $\mathbf{F}^\bullet$ by zero to all of $\mathcal{U}$. □

§5. The Microlocal Proof

We now return to the setting of Theorem 1.1. Let X be a complex analytic space embedded in an open subset $\mathcal{U}$ of $\mathbb{C}^{n+1}$, let $\mathcal{S} := \{S_\alpha\}$ be a complex Whitney stratification of X with connected strata, let $\mathbf{F}^\bullet \in D^b_{\mathcal{S}}(X)$, where the base ring is a p.i.d., and let $\hat{f} : \mathcal{U} \to \mathbb{C}$ be complex analytic. Let $f := \hat{f}_{|_X}$. Suppose that $\mathbf{0} \in X$ and $f(\mathbf{0}) = 0$.

If we let $j : X \hookrightarrow \mathcal{U}$ denote the inclusion, then the problem of studying $\mathbf{F}^\bullet$, f, X, and $\mathcal{S}$ is equivalent to studying $j_!\mathbf{F}^\bullet$, $\hat{f}$, $\mathcal{U}$, and $\mathcal{S} \cup \{\mathcal{U} - X\}$. **Therefore, throughout this section, we will assume without loss of generality, that $X = \mathcal{U}$, and so we write f in place of $\hat{f}$.**

We want to know how to alter the proof of Theorem 1.1 when we weaken our hypothesis, and only suppose that $\mathbf{0}$ is an isolated point in the support of $\phi_f\mathbf{F}^\bullet$, rather than being an isolated stratified critical point.

In fact, all we need are two lemmas: one which says the that a general complex linear perturbation of f has isolated stratified critical points, and one which says that any possible stratified critical points on the boundary of a small ball are irrelevant.

Lemma 5.1. *Let $L : X \to \mathbb{C}$ be a generic linear form. Then, there exists $\epsilon > 0$ such that, for all sufficiently small $\delta > 0$, for all $t \in \mathbb{D}^*_\delta$, $f + tL$ has isolated stratified critical points inside $\overset{\circ}{B}_\epsilon$ and no critical points which are contained in ∂B_ϵ.*

Proof. We believe that this is well-known – even in our case, where f may have stratified non-isolated critical points; however, for lack of a convenient reference in this generality, we supply a proof.

Fix a stratum $S_\alpha \neq \{\mathbf{0}\}$. Let L be generic enough so that $d_{\mathbf{0}}L \notin \left(\overline{T^*_{S_\alpha}X}\right)_{\mathbf{0}}$. Also, choose L so generic that the relative polar curve, $\Gamma^1_{f_{|\overline{S}_\alpha}, L}$, is purely 1-dimensional at the origin, and so that L is finite near the origin when restricted to the relative polar curve (that this is possible is well-known and is proved in many places; see, for instance, [**L-T**], 4.2.1 and [**Te**], 4.1.3.2.).

For any complex number $t \neq 0$, a critical point of $f + tL$ on S_α is either a critical point of $f_{|S_\alpha}$ or is a point on the relative polar curve. If we had an entire curve, containing the origin in its closure, of critical points of both $(f + tL)_{|S_\alpha}$ and $f_{|S_\alpha}$, then we would have $d_{\mathbf{0}}L \in \left(\overline{T^*_{S_\alpha}X}\right)_{\mathbf{0}}$ – a contradiction. Thus, any curve of critical points of $(f + tL)_{|S_\alpha}$ at the origin must be a component of $\Gamma^1_{f_{|\overline{S}_\alpha}, L}$. We will show that this can only happen for a finite number of "bad" t.

Let $\omega(u)$ be an analytic paramerization of a component of $\Gamma^1_{f_{|\overline{S}_\alpha}, L}$ such that $\omega(0) = \mathbf{0}$ and, for $u \neq 0$, $\omega(u) \in \Sigma(f + tL)_{|S_\alpha}$. Then, $d_{\omega(u)}(f + tL)(\omega'(u)) \equiv 0$ for $u \neq 0$. Therefore, $f(\omega(u)) + tL(\omega(u)) \equiv 0$, and so, as $L(\omega(u)) \not\equiv 0$ by our finiteness assumption, it follows that $\lim_{u\to 0} \dfrac{-f(\omega(u))}{L(\omega(u))}$ exists and is equal to t. Hence, there are at most as many "bad" t values as there are components of $\Gamma^1_{f_{|\overline{S}_\alpha}, L}$. The desired conclusion follows. $\square$

In the following lemma, we write $T^*_{V(f)}X$ in place of $\overline{T^*_{(V(f))_{\mathrm{reg}}}X}$. We also write Σf for the critical locus of f, which is unambiguous since we are assuming that the domain of f is an open subset of affine space. Finally, in some places below, we will identify T^*X with $X \times \mathbb{C}^{n+1} \cong X \times \mathbb{R}^{2n+2}$, so that we may use the real metric and minors of matrices.

Lemma 5.2. *Suppose that $\mathbf{0}$ is an isolated point in the support of $\phi_f\mathbf{F}^\bullet$. For complex numbers t and v, and linear forms L, let $\Psi_{L,t,v} : X \to \mathbb{R}$ be given by $\Psi_{L,t,v}(\mathbf{z}) := |f(\mathbf{z}) + tL(\mathbf{z}) - v|^2$.*

For all linear forms L, for all sufficiently small $\epsilon > 0$, if $j : B_\epsilon \hookrightarrow X$ denotes the inclusion, then there exists an open neighborhood $\mathcal{W}$ of $\partial B_\epsilon \cap V(f)$ in ∂B_ϵ and an open neighborhood $\mathcal{Y}$ of 0 in $\mathbb{C}$ such that, for all $(\mathbf{x}, t, v) \in \mathcal{W} \times \mathcal{Y} \times \mathbb{C}$, $(\mathbf{x}, d_{\mathbf{x}}\Psi_{L,t,v}) \notin SS(j_*j^*\mathbf{F}^\bullet)$ unless $\Psi_{L,t,v}(\mathbf{x}) = 0$.

Proof. Fix a linear form L. We will make a continuity argument to show that there exist such $\mathcal{W}$ and $\mathcal{Y}$ such that, for all $(\mathbf{x}, t) \in \mathcal{W} \times \mathcal{Y}$, for all $(a, b) \in \mathbb{R}^2 - \{\mathbf{0}\}$,

$$\big(\mathbf{x},\ a\, d_{\mathbf{x}}(\mathrm{Re}(f + tL)) + b\, d_{\mathbf{x}}(\mathrm{Im}(f + tL))\big) \notin SS(j_*j^*\mathbf{F}^\bullet).$$

This yields the desired result, since

$$d_{\mathbf{x}}\Psi_{L,t,v} \ = \ 2\,\mathrm{Re}(f(\mathbf{x}) + tL(\mathbf{x}) - v)\, d_{\mathbf{x}}(\mathrm{Re}(f + tL)) + 2\,\mathrm{Im}(f(\mathbf{x}) + tL(\mathbf{x}) - v)\, d_{\mathbf{x}}(\mathrm{Im}(f + tL)).$$

By Corollary 4.15, there exists $\epsilon_1 > 0$ such that, for all $\mathbf{x} \in B_{\epsilon_1} - \{\mathbf{0}\}$, $(\mathbf{x}, d_{\mathbf{x}}f) \notin SS(\mathbf{F}^\bullet)$, i.e., for all $\mathbf{x} \in (B_{\epsilon_1} - \{\mathbf{0}\}) \cap \operatorname{supp}\mathbf{F}^\bullet$, $d_{\mathbf{x}}f \notin \big(SS(\mathbf{F}^\bullet)\big)_{\mathbf{x}}$; in particular, for all $\mathbf{x} \in (B_{\epsilon_1} - \{\mathbf{0}\}) \cap \operatorname{supp}\mathbf{F}^\bullet$,

$d_{\mathbf{x}}f \neq 0$. By Remark 3.5, we conclude that, for all $\mathbf{x} \in (B_{\epsilon_1} - \{\mathbf{0}\}) \cap \operatorname{supp} \mathbf{F}^\bullet$, for all $(a,b) \in \mathbb{R}^2 - \{\mathbf{0}\}$, $a\, d_{\mathbf{x}}(\operatorname{Re} f) + b\, d_{\mathbf{x}}(\operatorname{Im} f) \notin \big(SS(\mathbf{F}^\bullet)\big)_{\mathbf{x}}$.

Now consider $SS(\mathbf{F}^\bullet) \mathbin{\widehat{+}} T^*_{V(f)}X$; this is a closed, $\mathbb{R}^+$-conic (actually, $\mathbb{C}$-conic), isotropic, subanalytic subset of T^*X. Let $r : X \to \mathbb{R}$ be given by $r(\mathbf{z}) := |\mathbf{z}|^2$. Then, the microlocal Bertini-Sard Theorem (4.3) implies that there exists $\epsilon_2 > 0$ such that, for all $\mathbf{x} \in r^{-1}(0, \epsilon_2^2)$,

$$(\dagger) \qquad (\mathbf{x}, d_{\mathbf{x}}r) \notin SS(\mathbf{F}^\bullet) \mathbin{\widehat{+}} T^*_{V(f)}X \ \supseteq\ SS(\mathbf{F}^\bullet)\ +\ T^*_{V(f)}X.$$

Let $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$. Let $U(SS(\mathbf{F}^\bullet))$ be the covectors of $SS(\mathbf{F}^\bullet)$ of unit length, i.e., $U(SS(\mathbf{F}^\bullet)) := \{(\mathbf{x}, \omega) \in SS(\mathbf{F}^\bullet) \mid \ ||\omega|| = 1\}$ (we have used the identification $T^*X \cong X \times \mathbb{R}^{2n+2}$). From the previous two paragraphs, we conclude that, for all $\mathbf{x} \in (B_{\epsilon_0} - \{\mathbf{0}\}) \cap \operatorname{supp} \mathbf{F}^\bullet$, for all $\omega \in \big(U(SS(\mathbf{F}^\bullet))\big)_{\mathbf{x}}$, the covectors $d_{\mathbf{x}}(\operatorname{Re} f)$, $d_{\mathbf{x}}(\operatorname{Im} f)$, $d_{\mathbf{x}}r$, and ω are linearly independent over $\mathbb{R}$.

Consider the function $M : U(SS(\mathbf{F}^\bullet)) \times \mathbb{C} \to X \times \mathbb{R}^k$ (for an appropriate k) given by

$$M(\mathbf{x}, \omega, t) := \left(\mathbf{x},\ 4 \times 4 \text{ minors of } \begin{pmatrix} d_{\mathbf{x}}(\operatorname{Re}(f + tL)) \\ d_{\mathbf{x}}(\operatorname{Im}(f + tL)) \\ d_{\mathbf{x}}r \\ \omega \end{pmatrix}\right).$$

Then, M is a continuous function, and, if $0 < \epsilon < \epsilon_0$,

$$M^{-1}(X \times \{\mathbf{0}\}) \ \cap\ \Big(\big[\big((\partial B_\epsilon \cap V(f)) \times \mathbb{R}^{2n+2}\big) \cap U(SS(\mathbf{F}^\bullet))\big] \times \{0\}\Big) \ =\ \emptyset.$$

By normality, it follows that there exists an open neighborhood $\mathcal{Z}$ of $\big[\big((\partial B_\epsilon \cap V(f)) \times \mathbb{R}^{2n+2}\big) \cap U(SS(\mathbf{F}^\bullet))\big] \times \{0\}$ in $U(SS(\mathbf{F}^\bullet)) \times \mathbb{C}$ such that, for all $(\mathbf{x}, \omega, t) \in \mathcal{Z}$, $d_{\mathbf{x}}(\operatorname{Re}(f + tL))$, $d_{\mathbf{x}}(\operatorname{Im}(f + tL))$, $d_{\mathbf{x}}r$, and ω are linearly independent over $\mathbb{R}$.

Using that $SS(\mathbf{F}^\bullet)$ is $\mathbb{R}^+$-conic and that $\partial B_\epsilon \cap V(f) \cap \operatorname{supp} \mathbf{F}^\bullet$ is compact, we immediately conclude that there exist an open neighborhood $\mathcal{W}'$ of $\partial B_\epsilon \cap V(f) \cap \operatorname{supp} \mathbf{F}^\bullet$ in ∂B_ϵ and an open neighborhood $\mathcal{Y}$ of 0 in $\mathbb{C}$ such that, for all $(\mathbf{x}, t) \in \mathcal{W}' \times \mathcal{Y}$, for all $(a,b) \in \mathbb{R}^2 - \{\mathbf{0}\}$,

$$(\ddagger) \qquad \big(\mathbf{x},\ a\, d_{\mathbf{x}}(\operatorname{Re}(f + tL)) + b\, d_{\mathbf{x}}(\operatorname{Im}(f + tL))\big) \ \notin\ SS(\mathbf{F}^\bullet) + T^*_{\partial B_\epsilon}X.$$

Let $\mathcal{W} := \mathcal{W}' \cup (\partial B_\epsilon - \operatorname{supp} \mathbf{F}^\bullet)$. As $SS(\mathbf{F}^\bullet) + T^*_{\partial B_\epsilon}X$ lies over $\operatorname{supp} \mathbf{F}^\bullet \cap \partial B_\epsilon$, $(\ddagger)$ also holds for all $(\mathbf{x}, t) \in \mathcal{W} \times \mathcal{Y}$ and $(a,b) \in \mathbb{R}^2 - \{\mathbf{0}\}$.

By $(\dagger)$ and Proposition 4.12, $SS(j_*j^*\mathbf{F}^\bullet) \subseteq SS(\mathbf{F}^\bullet) + N^*(B_\epsilon)$. In addition, if $\mathbf{x} \in \partial B_\epsilon$ and $(\mathbf{x}, \omega) \in SS(\mathbf{F}^\bullet) + N^*(B_\epsilon)$, then certainly $(\mathbf{x}, \omega) \in SS(\mathbf{F}^\bullet) + T^*_{\partial B_\epsilon}X$. The desired conclusion now follows from $(\ddagger)$. □

Finally, we can prove

Theorem 5.3. *Suppose that* $\mathbf{0}$ *is an isolated point in the support of* $\phi_f \mathbf{F}^\bullet$.

Then, for every $\mathbf{F}^\bullet$*-visible stratum* S_α, $(\mathbf{0}, d_{\mathbf{0}}f)$ *is an isolated point in* $\overline{T^*_{S_\alpha}X} \cap \operatorname{im} df$, *and if we let*

$$k_\alpha := \Big(\overline{T^*_{S_\alpha}X}\ \cdot\ \operatorname{im} df\Big)_{(\mathbf{0}, d_{\mathbf{0}}f)},$$

then

$$H^{i-1}(\phi_f\mathbf{F}^\bullet)_\mathbf{0} \;\cong\; \bigoplus_{\substack{\mathbf{F}^\bullet\text{-}visible\\ S_\alpha}} \big(R^{k_\alpha}\otimes_R \mathbb{H}^{i-d_\alpha}(\mathbb{N}_\alpha,\mathbb{L}_\alpha\;;\;\mathbf{F}^\bullet)\big).$$

Proof. Fix a linear form L and an $\epsilon > 0$ so that Lemmas 5.1 and 5.2 hold; let $\mathcal{W}$ and $\mathcal{Y}$ be as in Lemma 5.2.

If necessary, select ϵ smaller, so that all stratified critical points of f inside B_ϵ occur on $V(f)$. Also choose ϵ small enough, and a real $\omega_0 > 0$ small enough so that, if $0 < \omega \leqslant \omega_0$, $B_\epsilon \cap f^{-1}(\mathbb{D}_\omega)$ has hypercohomogy isomorphic to the stalk cohomology of $\mathbf{F}^\bullet$ at $\mathbf{0}$ and so that, for all $v \in \mathbb{D}^*_{\omega_0}$, $B_\epsilon \cap V(f-v)$ has the hypercohomology of the Milnor fibre. Also, choose ω_0 small enough so that $\partial B_\epsilon \cap f^{-1}(\mathbb{D}_{\omega_0}) \subseteq \mathcal{W}$. Fix a positive $\omega < \omega_0$.

Let $\mathbb{D}_\omega(v)$ denote the closed disk of radius ω, centered at v, and fix v so small that $B_\epsilon \cap f^{-1}(\mathbb{D}_\omega)$ has hypercohomogy isomorphic to that of $B_\epsilon \cap f^{-1}(\mathbb{D}_\omega(v))$; this is possible because all stratified critical points of f occur on $V(f)$. Again, by stratified Morse theory, for all small $t \in \mathcal{Y}$, if $\Psi_{L,t,v}(\mathbf{z}) := |f(\mathbf{z}) + tL(\mathbf{z}) - v|^2$, then $\Psi^{-1}_{L,t,v}[0,\omega^2]$ has the same hypercohomlogy as $B_\epsilon \cap f^{-1}(\mathbb{D}_\omega(v))$, and $\Psi^{-1}_{L,t,v}(0)$ has the same hypercohomlogy as $B_\epsilon \cap V(f-v)$. Now, apply microlocal Morse theory to $\Psi_{L,t,v}$ as its value goes from 0 to ω^2.

By Lemma 4.10, there is no change in hypercohomology as $\Psi_{L,t,v}$ goes from 0 to a sufficiently small positive value. After this, Lemma 4.9, combined with Lemma 5.2, guarantees that stratified critical points on ∂B_ϵ produce no change in hypercohomology. Hence, by Lemma 5.1, the remainder of the proof is precisely the same as that of Theorem 1.1. □

References

[**BBD**] A. Beilinson, J. Berstein, and P. Deligne, *Faisceaux Pervers*, Astérisque **100**, Soc. Math. de France, 1983.

[**BMM**] J. Briançon, P. Maisonobe, and M. Merle, *Localisation de systèmes différentiels, stratifications de Whitney et condition de Thom*, Invent. Math. **117** (1994), 531–550.

[**G**] V. Ginsburg, *Characteristic Varieties and Vanishing Cycles*, Inv. Math. **84** (1986), 327–403.

[**G-M1**] M. Goresky and R. MacPherson, *Intersection homology II*, Inv. Math **71** (1983), 77–129.

[**G-M2**] ______, *Stratified Morse Theory*, Ergebnisse der Math. 14, Springer-Verlag, Berlin, 1988.

[**K-S1**] M. Kashiwara and P. Schapira, *Micro-support des faisceaux*, C. R. Acad. Sci. **295** (1982), 487–490.

[**K-S2**] M. Kashiwara and P. Schapira, *Microlocal study of sheaves*, Astérisque 128, Soc. Math. France, 1985.

[**K-S3**] M. Kashiwara and P. Schapira, *Sheaves on Manifolds*, Grund. der math. Wiss. 292, Springer - Verlag, 1990.

[**L1**] Lê D. T., *Le concept de singularité isolée de fonction analytique*, Advanced Studies in Pure Math. **8** (1986), 215–227.

[**L2**] ______, *Morsification of* **D**-*Modules*, Bol. Soc. Mat. Mexicana (3) **4** (1998), 229–248.

[**L-T**] Lê D. T. and B. Teissier, *Variétés polaires locales et classes de Chern des variétiés singulières*, Annals of Math. **114** (1981).

[**M1**] D. Massey, *Critical Points of Functions on Singular Spaces*, Top. and Appl. **103** (2000), 55–93.

[**M2**] ______, *Hypercohomology of Milnor Fibres*, Topology **35** (1996), 969–1003.

[**M3**] ______, *The Sebastiani-Thom Isomorphism in the Derived Category*, Compos. Math. (to appear).

[**Sa**] C. Sabbah, *Quelques remarques sur la géométrie des espaces conormaux*, Astérisque **130** (1985), 161–192.

[**Si**] D. Siersma, *A bouquet theorem for the Milnor fibre*, J. Algebraic Geom. **4** (1995), 51–66.

[**Te**] B. Teissier, *Variétés polaires II: Multiplicités polaires, sections planes, et conditions de Whitney*, in Algebraic Geometry, Proc., La Rabida 1981, Springer Lect. Notes **961** (1982), 314–491.

[**Ti**] M. Tibăr, *Bouquet Decomposition of the Milnor Fibre*, Topology **35, no. 1** (1996), 227–241.

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