

Real Analytic Milnor Fibrations and a Strong Łojasiewicz Inequality*

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Abstract

We give a strong version of a classic inequality of Łojasiewicz; one which collapses to the usual inequality in the complex analytic case. We show that this inequality for real analytic functions allows us to construct a real Milnor fibration inside a ball.

1 Introduction

Suppose that \mathcal{U} is an open neighborhood of the origin in \mathbb{C}^N , and that $f_c : (\mathcal{U}, \mathbf{0}) \rightarrow (\mathbb{C}, 0)$ is a complex analytic function.

In the now-classic book [14], Milnor shows that one has what is now called the Milnor fibration of f_c (at $\mathbf{0}$). The Milnor fibration is THE fundamental device in the study of the topology of the hypersurface X defined by the vanishing of f_c .

In fact, there are two Milnor fibrations associated with f_c : one defined on small spheres, and one defined inside small open balls. Both of these are referred to as **the** Milnor fibration because the two fibrations are diffeomorphic. We wish to be precise.

For $\epsilon > 0$, let S_ϵ (resp., B_ϵ , \mathring{B}_ϵ) denote the sphere (resp., closed ball, open ball) of radius ϵ , centered at the origin. In the special case of balls in $\mathbb{R}^2 \cong \mathbb{C}$, we write \mathbb{D}_ϵ in place of B_ϵ and $\mathring{\mathbb{D}}_\epsilon$ in place of \mathring{B}_ϵ . Finally, we write \mathbb{D}_ϵ^* in place of $\mathbb{D}_\epsilon - \{\mathbf{0}\}$, and $\mathring{\mathbb{D}}_\epsilon^*$ in place of $\mathring{\mathbb{D}}_\epsilon - \{\mathbf{0}\}$.

One version of the Milnor fibration (at the origin) is given by: there exists $\epsilon_0 > 0$ such that, for all ϵ such that $0 < \epsilon \leq \epsilon_0$, the map $f_c/|f_c| : S_\epsilon - S_\epsilon \cap X \rightarrow S^1 \subseteq \mathbb{C}$ is a smooth, locally trivial fibration, whose diffeomorphism-type is independent of the choice of ϵ (see [14]).

The second, diffeomorphic version of the Milnor fibration is given by: there exists $\epsilon_0 > 0$ such that, for all ϵ such that $0 < \epsilon \leq \epsilon_0$, there exists a $\delta_0 > 0$ such that, for all δ such that $0 < \delta \leq \delta_0$, the map $f_c : \mathring{B}_\epsilon \cap f_c^{-1}(\partial\mathbb{D}_\delta) \rightarrow \partial\mathbb{D}_\delta \cong S^1 \subseteq \mathbb{C}$ is a smooth, locally trivial fibration, whose diffeomorphism-type is independent of the choice of ϵ and (sufficiently small) δ (see Theorem 5.11 of [14] and [9]). The primary advantage to this second characterization of the Milnor fibration is that it compactifies nicely to yield a locally trivial fibration $f_c : B_\epsilon \cap f_c^{-1}(\partial\mathbb{D}_\delta) \rightarrow \partial\mathbb{D}_\delta$, which, up to homotopy, is equivalent to either of the two previously-defined Milnor fibrations.

We shall not summarize the important properties of the Milnor fibration here, but refer the reader to [14], [2], [16], and the Introduction to [12]. We wish to emphasize that our discussion of the Milnor fibration above assumes that f_c is a complex analytic function.

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Of course, a complex analytic function yields a pair of real analytic functions, coming from the real and imaginary parts of the complex function, and one can ask the more general question: **when does a pair of real analytic functions possess one or both types of Milnor fibrations?**

This topic of real analytic Milnor fibrations is complicated and interesting, and gives rise to many questions.

In Chapter 11 of [14], Milnor discusses, fairly briefly, some results in the case of the real analytic function $f = (g, h)$. He considers the very special case where f has an isolated critical point at the origin, and shows that, while the restriction of f still yields a fibration over a small circle inside the ball, $f/|f|$ does not necessarily yield the projection map of a fibration from $S_\epsilon - S_\epsilon \cap X$ to S^1 ; see p. 99 of [14]. Can one relax the condition that f has an isolated critical point and still obtain a locally trivial fibration $f : {}^\circ B_\epsilon \cap f^{-1}(\partial \mathbb{D}_\delta) \rightarrow \partial \mathbb{D}_\delta$? Are there reasonable conditions that guarantee that $f/|f| : S_\epsilon - S_\epsilon \cap X \rightarrow S^1$ is a locally trivial fibration which is diffeomorphic to the fibration inside the ball? Such questions have been investigated by a number of researchers; see [14], [8], [15], [19], [18], and [3].

An obvious approach to answering the question about the existence of real analytic Milnor fibrations is simply to try to isolate what properties of a complex analytic function are used in proofs that Milnor fibrations exist. We write “proofs” and not “the proof” here because we will not follow Milnor’s proof, but rather follow Lê’s proof in [9] of the existence of Milnor fibrations inside the ball for complex analytic functions.

Lê’s proof consists almost solely of using the existence of a Thom (or a_f , or *good*) stratification. In [6], Hamm and Lê, following a suggestion of Pham, used the complex analytic Lojasiewicz inequality (see Corollary 1.3 below) and a “trick” to show that Thom stratifications exist.

Our goal in this paper is very modest: we will give the “correct” generalization of the complex analytic Lojasiewicz inequality, and then show that if a pair (or quadruple, or octuple) of real analytic functions satisfies this new *strong Lojasiewicz inequality*, then the Milnor fibration inside a ball exists.

In the remainder of the introduction, we will summarize our primary definition and result.

Let \mathcal{U} now denote an open subset of \mathbb{R}^n , and p denote a point in \mathcal{U} . Let g and h be real analytic functions from \mathcal{U} to \mathbb{R} , and let $f := (g, h) : \mathcal{U} \rightarrow \mathbb{R}^2$. Recall the classic inequality of Lojasiewicz [10] (see also [1]).

Theorem 1.1. *There exists an open neighborhood \mathcal{W} of p in \mathcal{U} , and $c, \theta \in \mathbb{R}$ such that $c > 0$, $0 < \theta < 1$, and, for all $x \in \mathcal{W}$,*

$$|g(x) - g(p)|^\theta \leq c |\nabla g(x)|,$$

where ∇g is the gradient vector.

Remark 1.2. The phrasing above is classical, and convenient in some arguments. However, one can also fix the value of the constant c above to be any $c > 0$, e.g., $c = 1$. In other words, one may remove the reference to c in the statement Theorem 1.1 and simply use the inequality

$$|g(x) - g(p)|^\theta \leq |\nabla g(x)|.$$

The argument is easy, and simply requires one to pick a larger θ (still less than 1). As we shall not use this “improved” statement, we leave the proof as an exercise.

Theorem 1.1 implies a well-known complex analytic version of itself. One can easily obtain this complex version by replacing g by the square of the norm of the complex analytic function. However, we shall prove two not so well-known more general corollaries, Corollary 3.1 and Corollary 3.2, which yield the complex analytic statement.

If n is even, say $n = 2m$, then we may consider the complexified version of f by defining f_c by

$$f_c(x_1 + iy_1, \dots, x_m + iy_m) := g(x_1, y_1, \dots, x_m, y_m) + ih(x_1, y_1, \dots, x_m, y_m).$$

From Corollary 3.2, we immediately obtain:

Corollary 1.3. *Suppose that f_c is complex analytic. Then, there exists an open neighborhood \mathcal{W} of p in $\mathcal{U} \subseteq \mathbb{C}^n$, and $c, \theta \in \mathbb{R}$ such that $c > 0$, $0 < \theta < 1$, and, for all $z \in \mathcal{W}$,*

$$|f_c(z) - f_c(p)|^\theta \leq c |\nabla f_c(z)|,$$

where ∇f_c denotes the complex gradient.

Our generalization of this complex analytic Lojasiewicz inequality is:

Definition 1.4. *We say that $f = (g, h)$ satisfies the **strong Lojasiewicz inequality at p** or that f is **L-analytic at p** if and only if there exists an open neighborhood \mathcal{W} of p in \mathcal{U} , and $c, \theta \in \mathbb{R}$ such that $c > 0$, $0 < \theta < 1$, and, for all $x \in \mathcal{W}$,*

$$|f(x) - f(p)|^\theta \leq c \cdot \min_{|(a,b)|=1} |a \nabla g(x) + b \nabla h(x)|.$$

Main Result. *Suppose that $f(\mathbf{0}) = \mathbf{0}$, that f is not locally constant near the origin, and that f is L-analytic at $\mathbf{0}$.*

Then, for all $0 < \delta \ll \epsilon \ll 1$, $f : B_\epsilon \cap f^{-1}(\partial \mathbb{D}_\delta) \rightarrow \partial \mathbb{D}_\delta$ is a proper, stratified submersion, and so $f : B_\epsilon \cap f^{-1}(\partial \mathbb{D}_\delta) \rightarrow \partial \mathbb{D}_\delta$ and $f : \circ B_\epsilon \cap f^{-1}(\partial \mathbb{D}_\delta) \rightarrow \partial \mathbb{D}_\delta$ are locally trivial fibrations.

Moreover, the diffeomorphism-type of $f : \circ B_\epsilon \cap f^{-1}(\partial \mathbb{D}_\delta) \rightarrow \partial \mathbb{D}_\delta$ is independent of the appropriately small choices of ϵ and δ .

We have presented the main definition and result above in the case of real analytic functions into \mathbb{R}^2 ; this was for simplicity of the discussion. In fact, in Definition 3.3 and Corollary 5.8, we give our main definition and result when the dimension of the codomain is arbitrary.

We thank T. Gaffney for pointing out the existence of nap-maps (see Definition 5.2) in all dimensions.

2 Singular Values of Matrices

In this section, we wish to recall some well-known linear algebra, and establish/recall some inequalities that we will need in later sections.

Let $\vec{v}_1, \dots, \vec{v}_k$ be vectors in \mathbb{R}^n . Let A denote the $n \times k$ matrix whose i -th column is \vec{v}_i . Let M denote the $k \times k$ matrix $A^t A$. Consider the function ν from the unit sphere centered at the origin in \mathbb{R}^k into the non-negative real numbers given by $\nu(t_1, \dots, t_k) := |t_1 \vec{v}_1 + \dots + t_k \vec{v}_k|$. The critical values of ν are the *singular values of A* , and they are the square roots of the necessarily non-negative eigenvalues of M . It is traditional to index the singular values in a decreasing manner, i.e., we let the singular values of A (which need not be distinct) be denoted by $\sigma_1, \dots, \sigma_k$, where $\sigma_1 \geq \dots \geq \sigma_k$. We denote the eigenvalues of M by $\lambda_i := \sigma_i^2$. The singular value σ_1 is the norm of A . The minimum singular value σ_k will be of particular interest throughout this paper. Note that

$$(\dagger) \quad \sigma_k = \min_{|(t_1, \dots, t_k)|=1} |t_1 \vec{v}_1 + \dots + t_k \vec{v}_k|.$$

Also note that the trace of M , $\text{tr}M$, is equal to $|\vec{v}_1|^2 + \dots + |\vec{v}_k|^2$, which is equal to $\lambda_1 + \dots + \lambda_k = \sigma_1^2 + \dots + \sigma_k^2$. Hence, $\text{tr}M = 0$ if and only if $A = 0$ if and only if $\sigma_1 = 0$. It will be important for us later that the singular values of A vary continuously with the entries of A . Also, as the non-zero eigenvalues of $A^t A$ and AA^t are the same, the non-zero singular values of A are equal to the non-zero singular values of A^t ; in particular, $\sigma_1(A) = \sigma_1(A^t)$, i.e., the norm of a matrix is equal to the norm of its transpose.

The following proposition contains the fundamental results on singular values that we shall need; these results are known, but we include brief proofs for the convenience of the reader.

Proposition 2.1.

1. $\sigma_k = 0$ if and only if $\vec{v}_1, \dots, \vec{v}_k$ are linearly dependent;

2. if $|\vec{v}_1|^2 + \dots + |\vec{v}_k|^2 \neq 0$, then

$$\frac{1}{\sqrt{k}} \leq \frac{\sigma_1}{\sqrt{|\vec{v}_1|^2 + \dots + |\vec{v}_k|^2}} \leq 1;$$

and

3. $k(\det M)^{1/k} \leq |\vec{v}_1|^2 + \dots + |\vec{v}_k|^2$, with equality holding if and only if $\vec{v}_1, \dots, \vec{v}_k$ all have the same length and are pairwise orthogonal.

Proof. Item 1 is immediate from (\dagger) .

Assume that $|\vec{v}_1|^2 + \dots + |\vec{v}_k|^2 \neq 0$. As $|\vec{v}_1|^2 + \dots + |\vec{v}_k|^2 = \sigma_1^2 + \dots + \sigma_k^2$, the right-hand inequality in Item 2 is immediate. For all i , let

$$r_i := \frac{\sigma_i}{\sqrt{|\vec{v}_1|^2 + \dots + |\vec{v}_k|^2}}.$$

Then, $r_1^2 + \dots + r_k^2 = 1$ and $r_1^2 \geq \dots \geq r_k^2$. It follows that $r_1^2 \geq 1/k$; this proves the left-hand inequality in Item 2.

The inequality in Item 3 is nothing more than the fact that the geometric mean of the eigenvalues of M is less than or equal to the arithmetic mean, where we have again used that the trace of M is $|\vec{v}_1|^2 + \dots + |\vec{v}_k|^2$. In addition, these two means are equal if and only if all of the eigenvalues of M are the same. This occurs if and only if all of singular values of A are the same, which would mean that the function $\nu(t_1, \dots, t_k) := |t_1 \vec{v}_1 + \dots + t_k \vec{v}_k|$ is constant on the unit sphere. The remainder of Item 3 follows easily. \square

We need some results for dealing with compositions. Let $\sigma_i(C)$ denote the i -th singular value of a matrix C , indexed in descending order. Let $\lambda_i(C) := \sigma_i^2(C)$, i.e., $\lambda_i(C)$ is the i -th eigenvalue of $C^t C$, where the eigenvalues are indexed in descending order.

Let B be an $m \times n$ matrix.

Proposition 2.2.

$$\sigma_k(BA) \geq \sigma_n(B)\sigma_k(A).$$

Hence,

$$[\det(A^t B^t BA)]^{1/k} \geq \sigma_k^2(BA) \geq \sigma_n^2(B)\sigma_k^2(A).$$

Proof. Let \vec{u} be a unit vector in \mathbb{R}^k , written as a $k \times 1$ matrix. If $A\vec{u} = 0$, then $\sigma_k(A) = 0$ and the first inequality is immediate. So, assume that $A\vec{u} \neq 0$. Then,

$$|BA\vec{u}| = \left| B \left(\frac{A\vec{u}}{|A\vec{u}|} \right) \right| \cdot |A\vec{u}| \geq \sigma_n(B)\sigma_k(A).$$

This proves the first inequality.

As $[\det(A^t B^t BA)]^{1/k}$ is the geometric mean of the $\sigma_i^2(BA)$, the second set of inequalities is immediate.

□

Lemma 2.3. *Suppose that P and Q are $r \times r$ real matrices, and that P is diagonalizable (over the reals) and has no negative eigenvalues. Then,*

$$\lambda_r(P)\text{tr}(Q) \leq \text{tr}(PQ) \leq \lambda_1(P)\text{tr}(Q).$$

Proof. Suppose that $P = S^{-1}DS$, where D is diagonal. Then, $\text{tr}(PQ) = \text{tr}(S^{-1}DSQ) = \text{tr}(DSQS^{-1})$. Let $B = SQS^{-1}$. Hence, $\text{tr}(B) = \text{tr}(Q)$, and

$$\text{tr}(PQ) = \text{tr}(DB) = \sum_{i=1}^r (DB)_{i,i} = \sum_{i=1}^r \sum_{j=1}^r D_{i,j}B_{j,i} = \sum_{i=1}^r D_{i,i}B_{i,i}.$$

As $\lambda_r(P) \leq D_{i,i} \leq \lambda_1(P)$, the conclusion follows. □

Proposition 2.4.

$$n\sigma_n^2(A^t)\sigma_n^2(B) \leq \text{tr}(A^t B^t BA) \leq n\sigma_1^2(A)\sigma_1^2(B).$$

Proof. As $\text{tr}(A^t B^t BA) = \text{tr}(AA^t B^t B)$, we may apply Lemma 2.3 with $P = A^t A$ and $Q = B^t B$, and conclude that

$$\sigma_n^2(A^t)[n\sigma_n^2(B)] \leq \sigma_n^2(A^t)\text{tr}(B^t B) \leq \text{tr}(A^t B^t BA) \leq \sigma_1^2(A^t)\text{tr}(B^t B) \leq \sigma_1^2(A^t)[n\sigma_1^2(B)].$$

Now, use that $\sigma_1(A) = \sigma_1(A^t)$. □

Corollary 2.5. *If $BA \neq 0$, then $\text{tr}(A^t B^t BA)$, $\sigma_1^2(B)$, and $\sigma_1^2(A)$ are not zero and*

$$\frac{k [\det(A^t B^t BA)]^{1/k}}{\text{tr}(A^t B^t BA)} \geq \frac{k\sigma_n^2(B)\sigma_k^2(A)}{n\sigma_1^2(B)\sigma_1^2(A)}.$$

Proof. Combine Proposition 2.4 with Proposition 2.2. □

3 L-maps and L-weights

As in the introduction, let \mathcal{U} be an open subset of \mathbb{R}^n . Let $f = (g, h)$ be a C^1 map from \mathcal{U} to \mathbb{R}^2 and let $G := (g_1, \dots, g_k)$ be a C^1 map from \mathcal{U} to \mathbb{R}^k . By the *critical locus of G* , ΣG , we mean the set of points where G is not a submersion (this is reasonable since our main hypothesis in most results will imply that $n \geq k$ or that G is locally constant). A number of our results will apply only with the stronger assumption that f and G are real analytic, but we shall state that hypothesis explicitly as needed.

Throughout this section, we let $A = A(x)$ denote the $n \times k$ matrix which has the gradient vector of $g_i(x)$ as its i -th column (i.e., A is the transpose of the derivative matrix $[d_x G]$ of (g_1, \dots, g_k)), and let $M := A^t A$. We will be applying the results of Section 2 to A and M . Let $\sigma_1(x), \dots, \sigma_k(x)$ denote the singular values of $A(x)$, indexed in decreasing order.

We have the following corollary to Theorem 1.1

Corollary 3.1. *Suppose that G is real analytic. Then, there exists an open neighborhood \mathcal{W} of p in \mathcal{U} , and $c, \theta \in \mathbb{R}$ such that $c > 0$, $0 < \theta < 1$, and, for all $x \in \mathcal{W}$,*

$$|G(x) - G(p)|^\theta \leq c \sqrt{|\nabla g_1(x)|^2 + \dots + |\nabla g_k(x)|^2}.$$

Proof. For notational convenience, we shall prove the result for $k = 2$, using $f = (g, h)$, and we shall assume that $f(p) = 0$; the proof of the general case proceeds in exactly the same manner.

We will prove that there exists \mathcal{W} , c , and θ as in the statement such that the inequality holds at all $x \in \mathcal{W}$ such that $f(x) \neq 0$. This clearly suffices to prove the corollary. We will also assume that $|\nabla g(p)|^2 + |\nabla h(p)|^2 = 0$ for, otherwise, the result is trivial.

Apply Theorem 1.1 to the function from $\mathcal{U} \times \mathcal{U}$ to \mathbb{R} given by $g^2(x) + h^2(w)$. We conclude that there exists an open neighborhood \mathcal{W} of p in \mathcal{U} , and $c, \theta \in \mathbb{R}$ such that $c > 0$, $0 < \theta < 1$, and, for all $(x, w) \in \mathcal{W} \times \mathcal{W}$,

$$|g^2(x) + h^2(w)|^\theta \leq c |2g(x)(\nabla g(x), 0) + 2h(w)(0, \nabla h(w))|.$$

Restricting to the diagonal, we obtain that, for all $x \in \mathcal{W}$,

$$|g^2(x) + h^2(x)|^\theta \leq 2c \sqrt{g^2(x)|\nabla g(x)|^2 + h^2(x)|\nabla h(x)|^2}.$$

Applying the Cauchy-Schwarz inequality, we have, for all $x \in \mathcal{W}$,

$$|g^2(x) + h^2(x)|^\theta \leq 2c \sqrt{|(g^2(x), h^2(x))| \cdot (|\nabla g(x)|^2 + |\nabla h(x)|^2)} = 2c (g^4(x) + h^4(x))^{1/4} (|\nabla g(x)|^4 + |\nabla h(x)|^4)^{1/4}.$$

For $a, b, x > 0$, $(a^x + b^x)^{1/x}$ is a decreasing function of x . Therefore, we conclude that, for all $x \in \mathcal{W}$,

$$|(g(x), h(x))|^{2\theta} = |g^2(x) + h^2(x)|^\theta \leq 2c (g^2(x) + h^2(x))^{1/2} (|\nabla g(x)|^2 + |\nabla h(x)|^2)^{1/2},$$

and so, for all $x \in \mathcal{W}$ such that $f(x) \neq 0$,

$$|f(x)|^{2\theta-1} \leq 2c \sqrt{|\nabla g(x)|^2 + |\nabla h(x)|^2}.$$

This proves the result, except for the bounds on the exponent. As $-1 < 2\theta - 1 < 1$, we have only to eliminate the possibility that $-1 < 2\theta - 1 \leq 0$. However, this is immediate, as we are assuming that $f(p) = 0$ and $|\nabla g(p)|^2 + |\nabla h(p)|^2 = 0$. \square

The following corollary follows at once.

Corollary 3.2. *Suppose that G is real analytic and, for all $x \in \mathcal{U}$, $\nabla g_1(x), \dots, \nabla g_k(x)$ have the same magnitude.*

Then, there exists an open neighborhood \mathcal{W} of p in \mathcal{U} , and $c, \theta \in \mathbb{R}$ such that $c > 0$, $0 < \theta < 1$, and, for all $x \in \mathcal{W}$,

$$|G(x) - G(p)|^\theta \leq c |\nabla g_1(x)|.$$

We now give the fundamental definition of this paper. Our intention is to isolate the properties of a complex analytic function that are used in proving the existence of the Milnor fibration inside a ball.

Definition 3.3. *We say that G satisfies the **strong Łojasiewicz inequality** at p or is an **L-map** at p if and only if there exists an open neighborhood \mathcal{W} of p in \mathcal{U} , and $c, \theta \in \mathbb{R}$ such that $c > 0$, $0 < \theta < 1$, and, for all $x \in \mathcal{W}$,*

$$|G(x) - G(p)|^\theta \leq c \cdot \min_{|(a_1, \dots, a_k)|=1} |a_1 \nabla g_1(x) + \dots + a_k \nabla g_k(x)| = c \sigma_k(x).$$

*If G satisfies the strong Łojasiewicz inequality at p and is real analytic in a neighborhood of p , then we say that G is **L-analytic** at p .*

*We say that G is **L-analytic** if and only if G is L-analytic at each point $p \in \mathcal{U}$.*

Remark 3.4. Note that if $\nabla g_1(x), \dots, \nabla g_k(x)$ are always pairwise orthogonal and have the same length, then the inequality in Definition 3.3 collapses to

$$|G(x) - G(p)|^\theta \leq c |\nabla g_1(x)|,$$

which, as we saw in Corollary 3.2, is automatically satisfied if G is real analytic.

In the case $k = 2$, and $f = (g, h)$, one easily calculates the eigenvalues of the matrix

$$\begin{bmatrix} |\nabla g|^2 & \nabla g \cdot \nabla h \\ \nabla g \cdot \nabla h & |\nabla h|^2 \end{bmatrix}$$

and finds that $\sigma_2(x)$ is

$$\sqrt{\frac{|\nabla g(x)|^2 + |\nabla h(x)|^2 - \sqrt{(|\nabla g(x)|^2 + |\nabla h(x)|^2)^2 - 4[|\nabla g(x)|^2 |\nabla h(x)|^2 - (\nabla g(x) \cdot \nabla h(x))^2]}}{2}},$$

Definition 3.5. *When G is such that $\nabla g_1(x), \dots, \nabla g_k(x)$ are always pairwise orthogonal and have the same length, we say that G is a **simple L-map**.*

Thus, G is a simple L-map if and only if M is a scalar multiple of the identity, i.e., for all $x \in \mathcal{U}$, $M(x) = |\nabla g_1(x)|^2 I_k$.

Remark 3.6. Note that the pair of functions given by the real and imaginary parts of a holomorphic or anti-holomorphic function yield a simple L-analytic map.

Definition 3.7. At each point $x \in \mathcal{U}$ where $d_x G \neq 0$ (i.e., where $|\nabla g_1(x)|^2 + \cdots + |\nabla g_k(x)|^2 \neq 0$), define

$$\rho_G(x) := \frac{k (\det M)^{1/k}}{\operatorname{tr} M} = \frac{k (\det M)^{1/k}}{|\nabla g_1(x)|^2 + \cdots + |\nabla g_k(x)|^2}.$$

If $d_x G$ is not identically zero, we define the **L-weight** of G , ρ_G^{\inf} , to be the infimum of $\rho_G(x)$ over all $x \in \mathcal{U}$ such that $|\nabla g_1(x)|^2 + \cdots + |\nabla g_k(x)|^2 \neq 0$.

If $d_x G$ is identically zero, so that G is locally constant, we set ρ_G^{\inf} equal to 1.

We say that G **has positive L-weight** at x if and only if there exists an open neighborhood \mathcal{W} of x such that $G|_{\mathcal{W}}$ has positive L-weight.

Remark 3.8. If G is a submersion at x , then it is clear that G has positive L-weight at x . However, the converse is not true; for instance, a simple L-map has positive L-weight (see below). Thus, in a sense, having positive L-weight is a generalization of being a submersion.

Note that, if $\rho_G^{\inf} > 0$, we must have either $n \geq k$ or that G is locally constant; furthermore, it must necessarily be true that $x \in \Sigma G$ implies that $\nabla g_1(x) = \cdots = \nabla g_k(x) = \mathbf{0}$.

For a pair of functions $f = (g, h)$, one easily calculates that

$$\rho_f(x) = \frac{2|\nabla g(x)||\nabla h(x)|\sin \eta(x)}{|\nabla g(x)|^2 + |\nabla h(x)|^2},$$

where $\eta(x)$ equals the angle between $\nabla g(x)$ and $\nabla h(x)$.

One immediately concludes from Proposition 2.1:

Proposition 3.9. Suppose that $d_x G \neq 0$. Then,

1. $0 \leq \rho_G(x) \leq 1$;
2. $\rho_G(x) = 0$ if and only if $\nabla g_1(x), \dots, \nabla g_k(x)$ are linearly dependent;
3. $\rho_G(x) = 1$ if and only if $\nabla g_1(x), \dots, \nabla g_k(x)$ all have the same length and are pairwise orthogonal.

Thus, $0 \leq \rho_G^{\inf} \leq 1$, and $\rho_G^{\inf} = 1$ if and only if G is a simple L-map.

In the case where $k = 2$, there is a very precise relation between $\rho_f(x)$ and the strong L-inequality.

Proposition 3.10. There exist positive constants α and β such that, at all points $x \in \mathcal{U}$ such that $d_x f \neq 0$,

$$\alpha \sigma_2(x) \leq \rho_f(x) \sqrt{|\nabla g(x)|^2 + |\nabla h(x)|^2} \leq \beta \sigma_2(x).$$

Hence, f satisfies the strong Lojasiewicz inequality at p if and only if there exists an open neighborhood \mathcal{W} of p in \mathcal{U} , and $c, \theta \in \mathbb{R}$ such that $c > 0$, $0 < \theta < 1$, and, for all $x \in \mathcal{W}$ such that $d_x f \neq 0$,

$$|f(x) - f(p)|^\theta \leq c \rho_f(x) \sqrt{|\nabla g(x)|^2 + |\nabla h(x)|^2},$$

and, for all $x \in \mathcal{W}$ such that $d_x f = 0$, $f(x) = f(p)$.

Proof. Assume that $d_x f \neq 0$. Then, the first set of inequalities is trivially true if $\sigma_2(x) = 0$; so, assume $\sigma_2(x) \neq 0$. We need to show that we can find $\alpha, \beta > 0$ such that

$$\alpha \leq \frac{\rho_f(x) \sqrt{|\nabla g(x)|^2 + |\nabla h(x)|^2}}{\sigma_2(x)} \leq \beta.$$

This follows immediately from Item 2 of Proposition 2.1 since

$$\frac{\rho_f(x) \sqrt{|\nabla g(x)|^2 + |\nabla h(x)|^2}}{\sigma_2(x)} = \frac{2\sqrt{\sigma_1^2(x)\sigma_2^2(x)}}{\sigma_2(x) \sqrt{|\nabla g(x)|^2 + |\nabla h(x)|^2}} = \frac{2\sigma_1(x)}{\sqrt{|\nabla g(x)|^2 + |\nabla h(x)|^2}}.$$

□

Proposition 3.11. *The following are equivalent:*

1. $\rho_G^{\text{inf}} > 0$;
2. there exists $b > 0$ such that, for all $x \in \mathcal{U}$, $b \cdot \sqrt{|\nabla g_1(x)|^2 + \cdots + |\nabla g_k(x)|^2} \leq \sigma_k(x)$;
3. there exists $b > 0$ such that, for all $x \in \mathcal{U}$, $b \cdot \sigma_1(x) \leq \sigma_k(x)$.

Proof. Throughout this proof, when we take an infimum, we mean the infimum over all $x \in U$ such that $d_x G \neq 0$; this implies that $\sigma_1(x) \neq 0$.

Note that Items 2 and 3 simply say that

$$0 < \inf \frac{\sigma_k(x)}{\sqrt{|\nabla g_1(x)|^2 + \cdots + |\nabla g_k(x)|^2}}$$

and

$$0 < \inf \frac{\sigma_k(x)}{\sigma_1(x)}.$$

Now,

$$\begin{aligned} \rho_G(x) &= \frac{k(\det M)^{1/k}}{|\nabla g_1(x)|^2 + \cdots + |\nabla g_k(x)|^2} = \frac{k(\sigma_1^2(x) \cdots \sigma_k^2(x))^{1/k}}{|\nabla g_1(x)|^2 + \cdots + |\nabla g_k(x)|^2} = \\ &k \left[\frac{\sigma_1(x)}{\sqrt{|\nabla g_1(x)|^2 + \cdots + |\nabla g_k(x)|^2}} \cdots \frac{\sigma_k(x)}{\sqrt{|\nabla g_1(x)|^2 + \cdots + |\nabla g_k(x)|^2}} \right]^{2/k}, \end{aligned}$$

and the factors are non-negative, at most 1, and are in decreasing order. This immediately yields the equivalence of Items 1 and 2.

The equivalence of Items 2 and 3 follows at once from

$$\begin{aligned} \frac{\sigma_k(x)}{\sqrt{|\nabla g_1(x)|^2 + \cdots + |\nabla g_k(x)|^2}} &= \frac{\sigma_k(x)}{\sqrt{\sigma_1^2(x) + \cdots + \sigma_k^2(x)}} = \\ &\frac{\sigma_k(x)/\sigma_1(x)}{\sqrt{1 + (\sigma_2(x)/\sigma_1(x))^2 + \cdots + (\sigma_k(x)/\sigma_1(x))^2}}. \end{aligned}$$

□

Our primary interest in \mathbf{L} -weights is due to the following:

Theorem 3.12. *Suppose that G is real analytic and $\rho_G^{\text{inf}} > 0$. Then, G is L -analytic.*

Proof. By Corollary 3.1, if x is near p and $d_x G = 0$, then $G(x) = G(p)$. Thus, it suffices to verify that the strong Łojasiewicz inequality holds near any point at which $d_x G \neq 0$. The desired inequality follows at once from Item 2 of Proposition 3.11, combined with Corollary 3.1. \square

Let $H := (h_1, \dots, h_n)$ be a C^1 map from an open subset \mathcal{W} of \mathbb{R}^m to \mathcal{U} . For $x \in \mathcal{W}$, we let $A = A(H(x))$ be the matrix $[d_{H(x)}G]^t$, and set $B = B(x) := [d_x H]^t$. Let $C = C(x)$ denote the matrix $[d_x(G \circ H)]^t = BA$.

Theorem 3.13. *Suppose that $q \in \mathcal{W}$, that H which has positive L -weight at q , and that G has positive L -weight at $H(q)$. Then, $G \circ H$ has positive L -weight at q .*

Proof. At a point $x \in \mathcal{W}$ where $C \neq 0$, Corollary 2.5 tells us that

$$\rho_{G \circ H}(x) \geq \frac{k\sigma_n^2(B(x))\sigma_k^2(A(H(x)))}{n\sigma_1^2(B(x))\sigma_1^2(A(H(x)))}.$$

By Item 3 of Proposition 3.11, there exists an open neighborhood \mathcal{W}' of q on which the infimum b_H of $\sigma_n(B)/\sigma_1(B)$ is positive, and an open neighborhood \mathcal{U}' of $H(q)$ on which the infimum b_G of $\sigma_n(A)/\sigma_1(A)$ is positive. Therefore, the infimum of $\rho_{G \circ H}(x)$ over all $x \in \mathcal{W}' \cap H^{-1}(\mathcal{U}')$ such that $C(x) \neq 0$ is at least $kb_H^2 b_G^2/n$. \square

Theorem 3.13 gives us an easy way of producing new non-simple L -analytic maps: take a map into \mathbb{R}^2 consisting of the real and imaginary parts of a holomorphic or anti-holomorphic function, and then compose with a real analytic change of coordinates on an open set in \mathbb{R}^2 . Actually, we can give a much more precise result when H is a simple L -map.

Proposition 3.14. *Suppose that H is a simple L -map. Then, for all $x \in \mathcal{W}$ at which $d_x(G \circ H) \neq 0$, $\rho_{G \circ H}(x) = \rho_G(H(x))$.*

Proof. Suppose that $d_x(G \circ H) \neq 0$. As H is a simple L -map, $B^t B = \lambda I_n$, where $\lambda = |\nabla h_1(x)|^2$. Thus,

$$\rho_{G \circ H}(x) = \frac{k[\det(A^t B^t B A)]^{1/k}}{\text{tr}(A^t B^t B A)} = \frac{k[\lambda^k \det(A^t A)]^{1/k}}{\lambda \text{tr}(A^t A)} = \frac{k[\det(A^t A)]^{1/k}}{\text{tr}(A^t A)} = \rho_G(H(x)).$$

\square

We give two quick examples of how to produce interesting L -analytic maps.

Example 3.15. As we mentioned above, if g and h are the real and imaginary parts of a holomorphic or anti-holomorphic function, then $f = (g, h)$ is a simple L -analytic function.

One can also mix holomorphic and anti-holomorphic functions. Let $z = x + iy$ and $w = u + iv$, and consider the real and imaginary parts of $\bar{z}w^2$, i.e., let $g = x(u^2 - v^2) + 2yuv$ and $h = 2xuv - y(u^2 - v^2)$. It is trivial to verify that (g, h) is simple L -analytic.

More generally, if (g_1, h_1) and (g_2, h_2) are simple L -analytic, then the real and imaginary parts of

$$(g_1(\mathbf{z}) + ih_1(\mathbf{z}))(g_2(\mathbf{w}) + ih_2(\mathbf{w}))$$

will yield a new simple L -analytic function.

Example 3.16. Suppose that $f = (g, h)$ is a simple \mathbb{L} -analytic map.

Consider the linear map L of \mathbb{R}^2 given by $(u, v) \mapsto (au + bv, cu + dv)$. Then, at every point, the matrix of the derivative of L is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and one calculates that ρ_L is constantly $2|ad - bc|/(a^2 + b^2 + c^2 + d^2)$. Thus, if $ad - bc \neq 0$, then $\rho_L > 0$ and, unless $ab + cd = 0$ and $a^2 + c^2 = b^2 + d^2$, we have that $\rho_L < 1$.

Therefore, if L is an isomorphism which is not an orthogonal transformation composed with scalar multiplication, then Proposition 3.14 tells us that $(ag + bh, cg + dh)$ has positive \mathbb{L} -weight less than one and, hence, is an \mathbb{L} -analytic map which is not simple and, therefore, cannot arise from a holomorphic or anti-holomorphic complex function.

Relationship to Jacquemard's Conditions

For the remainder of this section, we will restrict ourselves to considering a real analytic map $f = (g, h)$ into \mathbb{R}^2 . In [8], Jacquemard investigates such f satisfying three conditions. We will refer to these conditions as $J0$, $J1$ and $J2$; actually, our condition $J2$ will be the weaker condition given by Ruas, Seade, and Verjovsky in [17].

Definition 3.17. *The Jacquemard conditions are:*

($J0$) *the origin is an isolated critical point of f ;*

($J1$) *there exists an open neighborhood \mathcal{W} of 0 in \mathcal{U} and a real number $\tau > 0$ such that, if $x \in \mathcal{W}$ is such that $\nabla g(x) \neq \mathbf{0}$ and $\nabla h(x) \neq \mathbf{0}$, then*

$$\left| \frac{\nabla g(x)}{|\nabla g(x)|} \cdot \frac{\nabla h(x)}{|\nabla h(x)|} \right| \leq 1 - \tau;$$

($J2$) *the real integral closures of the Jacobian ideals of g and h , inside the ring of real analytic germs at the origin, are equal.*

The reader should note that we may discuss the Jacquemard conditions holding independently; in particular, we will assume in some settings that $J1$ and $J2$ hold, without assuming $J0$.

The importance of the Jacquemard conditions stems from the following theorem, which is essentially proved in [8], and improved using $J2$ in [17].

Theorem 3.18. *If Jacquemard's conditions hold, then f satisfies the strong Milnor condition, i.e., there exists $\epsilon_0 > 0$ such that, for all ϵ such that $0 < \epsilon \leq \epsilon_0$, the map $f/|f|$ from S_ϵ (the sphere of radius ϵ , centered at the origin) to \mathbb{R}^2 is a smooth, locally trivial fibration.*

We wish to see that, if we assume $J1$, greatly weaken $J2$, and omit $J0$, then f has positive \mathbb{L} -weight at 0 and, hence, is \mathbb{L} -analytic at 0. It will then follow from Corollary 5.8 that the Milnor fibration inside a ball, centered at 0, exists. First, we need a lemma.

Lemma 3.19. *Condition $J2$ implies that there exists an open neighborhood \mathcal{W} of the origin in \mathcal{U} in which ∇g and ∇h are comparable in magnitude, i.e., such that there exist $A, B > 0$ such that, for all $x \in \mathcal{W}$,*

$$A|\nabla g(x)| \leq |\nabla h(x)| \leq B|\nabla g(x)|.$$

Proof. By Proposition 4.2 of [5], J2 is equivalent to: there exists a neighborhood \mathcal{W} of the origin in \mathcal{U} and $C_1, C_2 > 0$ such that, at all points in \mathcal{W} , for all i such that $1 \leq i \leq n$, $|\partial g/\partial x_i| \leq C_1 \max_j |\partial h/\partial x_j|$ and $|\partial h/\partial x_i| \leq C_2 \max_j |\partial g/\partial x_j|$. One quickly concludes that, at all points of \mathcal{W} , $|\nabla g| \leq \sqrt{n}C_1|\nabla h|$ and $|\nabla h| \leq \sqrt{n}C_2|\nabla g|$. The lemma follows. \square

Proposition 3.20. *Suppose there exists an open neighborhood \mathcal{W} of the origin in \mathcal{U} in which ∇g and ∇h are comparable in magnitude. Then, Condition J1 is satisfied if and only if f has positive L-weight at 0.*

Proof. First, note that, when ∇g and ∇h are comparable in magnitude, then one of them equals zero at a point x if and only if both of them equal zero at x . Suppose that $A, B > 0$ are such that, for all $x \in \mathcal{W}$,

$$A|\nabla g(x)| \leq |\nabla h(x)| \leq B|\nabla g(x)|.$$

Let $\eta(x)$ equal the angle between $\nabla g(x)$ and $\nabla h(x)$. Then,

$$\left| \frac{\nabla g(x)}{|\nabla g(x)|} \cdot \frac{\nabla h(x)}{|\nabla h(x)|} \right| = |\cos \eta(x)|,$$

and it is trivial to conclude that Condition J1 holds if and only if there exists an open neighborhood of the origin $\mathcal{W}' \subseteq \mathcal{W}$ such that $0 < \inf_{x \in \mathcal{W}'} (\sin \eta(x))$.

Recall from Remark 3.8 that, at a point x where $|\nabla g(x)|^2 + |\nabla h(x)|^2 \neq 0$,

$$\rho_f(x) = \frac{2|\nabla g(x)||\nabla h(x)| \sin \eta(x)}{|\nabla g(x)|^2 + |\nabla h(x)|^2}.$$

Now, $A|\nabla g(x)| \leq |\nabla h(x)| \leq B|\nabla g(x)|$ implies that

$$\frac{2A}{1+B^2} \leq \frac{2|\nabla g(x)||\nabla h(x)|}{|\nabla g(x)|^2 + |\nabla h(x)|^2} \leq \frac{2B}{1+A^2}.$$

The conclusion follows. \square

Remark 3.21. In light of Lemma 3.19 and Proposition 3.20, we see that the Jacquemard conditions are far stronger than what one needs to conclude that f has positive L-weight at the origin.

4 Milnor's conditions (a) and (b)

In this section, we will discuss general conditions which allow us to conclude that Milnor fibrations exist.

We continue with our notation from the previous section, except that we now assume that G is only C^∞ . For notational convenience, we restrict our attention to the case where the point p is the origin and where $G(\mathbf{0}) = \mathbf{0}$. We also assume that G is not locally constant near $\mathbf{0}$. Let $X := G^{-1}(\mathbf{0}) = V(G)$.

Let $\mathfrak{A} = \Sigma G$, so that \mathfrak{A} is the closed set of points in \mathcal{U} at which the gradients $\nabla g_1, \nabla g_2, \dots, \nabla g_k$ are linearly dependent. Let r denote the function given by the square of the distance from the origin, and let \mathfrak{B} denote the closed set of points in \mathcal{U} at which the gradients $\nabla r, \nabla g_1, \nabla g_2, \dots, \nabla g_k$ are linearly dependent. Of course, $\mathfrak{A} \subseteq \mathfrak{B}$.

We wish to give names to the submersive conditions necessary to apply Ehresmann's Theorem [4] (in the case of manifolds with boundary).

Definition 4.1. We say that the map G satisfies **Milnor's condition (a) at $\mathbf{0}$** (or that $\mathbf{0}$ is an **isolated critical value of G near $\mathbf{0}$**) if and only if $\mathbf{0} \notin \overline{\mathfrak{A} - X}$, i.e., if $\Sigma G \subseteq V(G)$ near $\mathbf{0}$.

We say that the map G satisfies **Milnor's condition (b) at $\mathbf{0}$** if and only if $\mathbf{0}$ is an isolated point of (or, is not in) $X \cap \overline{\mathfrak{B} - X}$.

If G satisfies Milnor's condition (a) (respectively, (b)), then we say that $\epsilon > 0$ is a **Milnor (a) (respectively, (b)) radius for G at $\mathbf{0}$** provided that $B_\epsilon \cap (\overline{\mathfrak{A} - X}) = \emptyset$ (respectively, $B_\epsilon \cap X \cap (\overline{\mathfrak{B} - X}) \subseteq \{\mathbf{0}\}$). We say simply that $\epsilon > 0$ is a **Milnor radius for G at $\mathbf{0}$** if and only if ϵ is both a Milnor (a) and Milnor (b) radius for G at $\mathbf{0}$.

Remark 4.2. Using our notation from the introduction, if f_c is complex analytic, then $f = (g, h)$ satisfies Milnor's conditions (a) and (b); in this case, Milnor's condition (a) is well-known and follows easily from a curve selection argument or from Corollary 1.3, and Milnor's condition (b) follows from the existence of good (or a_f) stratifications of $V(f)$ (see [6] and [9], and below).

Lemma 4.3. Suppose that the map G satisfies Milnor's condition (b) at $\mathbf{0}$ and that $\epsilon_0 > 0$ is a Milnor (b) radius for G at $\mathbf{0}$. Then, for all ϵ such that $0 < \epsilon < \epsilon_0$, there exists $\delta_\epsilon > 0$ such that the map

$$H : (\mathring{B}_{\epsilon_0} - B_\epsilon) \cap G^{-1}(\mathring{B}_{\delta_\epsilon} - \{\mathbf{0}\}) \rightarrow (\mathring{B}_{\delta_\epsilon} - \{\mathbf{0}\}) \times (\epsilon^2, \epsilon_0^2)$$

given by $H(x) = (G(x), |x|^2)$ is a proper submersion.

In particular, for all ϵ' such that $0 < \epsilon' < \epsilon_0$, there exists $\delta_{\epsilon'} > 0$ such that

$$G : \partial B_{\epsilon'} \cap G^{-1}(\mathring{B}_{\delta_{\epsilon'}} - \{\mathbf{0}\}) \rightarrow \mathring{B}_{\delta_{\epsilon'}} - \{\mathbf{0}\}$$

is a proper submersion.

Proof. That H is proper is easy. Let $\pi : (\mathring{B}_{\delta_\epsilon} - \{\mathbf{0}\}) \times (\epsilon^2, \epsilon_0^2) \rightarrow (\epsilon^2, \epsilon_0^2)$ denote the projection. Suppose that $C \subseteq (\mathring{B}_{\delta_\epsilon} - \{\mathbf{0}\}) \times (\epsilon^2, \epsilon_0^2)$ is compact. Then, $\pi(C)$ is compact, and $H^{-1}(C)$ is a closed subset of the compact set $\{x \in \mathring{B}_{\epsilon_0} \mid |x|^2 \in \pi(C)\}$. Thus, $H^{-1}(C)$ is compact.

Now, a critical point of H is precisely a point in $\mathfrak{B} \cap (\mathring{B}_{\epsilon_0} - B_\epsilon) \cap G^{-1}((\mathring{B}_{\delta_\epsilon} - \{\mathbf{0}\}) \times (\epsilon^2, \epsilon_0^2))$. Suppose that had such a point, regardless of how small we choose $\delta_\epsilon > 0$. Then, we would have a sequence $x_i \in (\mathfrak{B} - X) \cap (\mathring{B}_{\epsilon_0} - B_\epsilon)$ such that $G(x_i) \rightarrow 0$. As the x_i are in the compact set $B_{\epsilon_0} - \mathring{B}_\epsilon$, by taking a subsequence if necessary, we may assume that $x_i \rightarrow x \in B_{\epsilon_0} - \mathring{B}_\epsilon$. As $G(x_i) \rightarrow 0$, $x \in X$. Therefore, $x \in (B_{\epsilon_0} - \mathring{B}_\epsilon) \cap X \cap \overline{(\mathfrak{B} - X)}$; a contradiction, since ϵ_0 is a Milnor (b) radius. Hence, H is a submersion.

The last statement follows at once from this, since one need only pick an ϵ such that $0 < \epsilon < \epsilon'$ and apply that H is a submersion. \square

Theorem 4.4. Suppose that G satisfies Milnor's conditions (a) and (b) at $\mathbf{0}$, and let ϵ_0 be a Milnor radius for G at $\mathbf{0}$.

Then, for all ϵ such that $0 < \epsilon < \epsilon_0$, there exists $\delta_\epsilon > 0$ such that the map

$$G : B_\epsilon \cap G^{-1}(\mathring{B}_{\delta_\epsilon} - \{\mathbf{0}\}) \rightarrow \mathring{B}_{\delta_\epsilon} - \{\mathbf{0}\}$$

is a proper, stratified submersion and, hence, a locally trivial fibration, in which the local trivializations preserve the strata.

In addition, for all such $(\epsilon, \delta_\epsilon)$ pairs, for all δ such that $0 < \delta < \delta_\epsilon$, the map

$$G : B_\epsilon \cap G^{-1}(\partial B_\delta) \rightarrow \partial B_\delta$$

is a proper, stratified submersion and, hence, a locally trivial fibration, whose diffeomorphism-type is independent of the choice of such ϵ and δ .

It follows that, for all such $(\epsilon, \delta_\epsilon)$ pairs, for all δ such that $0 < \delta < \delta_\epsilon$, the map

$$G : \circ B_\epsilon \cap G^{-1}(\partial B_\delta) \rightarrow \partial B_\delta$$

is a locally trivial fibration, whose diffeomorphism-type is independent of the choice of such ϵ and δ .

Proof. The map $G : B_\epsilon \cap G^{-1}(\circ B_{\delta_\epsilon} - \{\mathbf{0}\}) \rightarrow \circ B_{\delta_\epsilon} - \{\mathbf{0}\}$ is clearly proper.

Milnor's condition (a) tells us that

$$G : \circ B_\epsilon \cap G^{-1}(\circ B_{\delta_\epsilon} - \{\mathbf{0}\}) \rightarrow \circ B_{\delta_\epsilon} - \{\mathbf{0}\},$$

is a submersion regardless of the choice of $\delta_\epsilon > 0$.

The last line of Lemma 4.3 tells us that we may pick $\delta_\epsilon > 0$ such that the map

$$G : \partial B_\epsilon \cap G^{-1}(\circ B_{\delta_\epsilon} - \{\mathbf{0}\}) \rightarrow \circ B_{\delta_\epsilon} - \{\mathbf{0}\}$$

is a submersion.

Therefore,

$$G : B_\epsilon \cap G^{-1}(\circ B_{\delta_\epsilon} - \{\mathbf{0}\}) \rightarrow \circ B_{\delta_\epsilon} - \{\mathbf{0}\}$$

is a proper, stratified submersion, and so, by Ehresmann's Theorem (with boundary) [4] or Thom's first isotopy lemma [13], this map is a locally trivial fibration, in which the local trivializations preserve the strata.

It follows at once that for a fixed such ϵ , for all δ such that $0 < \delta < \delta_\epsilon$, the map

$$G : B_\epsilon \cap G^{-1}(\partial B_\delta) \rightarrow \partial B_\delta$$

is a proper, stratified submersion and, hence, a locally trivial fibration, whose diffeomorphism-type is independent of the choice of such δ .

It remains for us to show that, if we pick $0 < \epsilon < \epsilon' < \epsilon_0$, then there exists $\delta > 0$ such that $\delta < \min\{\delta_\epsilon, \delta_{\epsilon'}\}$, and such that the fibrations $G : B_\epsilon \cap G^{-1}(\partial B_\delta) \rightarrow \partial B_\delta$ and $G : B_{\epsilon'} \cap G^{-1}(\partial B_\delta) \rightarrow \partial B_\delta$ have the same diffeomorphism-type.

Let $\hat{\epsilon}$ be such that $0 < \hat{\epsilon} < \epsilon$, let $\delta_{\hat{\epsilon}} > 0$ be as in the first part of Lemma 4.3, and let $\delta > 0$ be less than $\min\{\delta_\epsilon, \delta_{\epsilon'}, \delta_{\hat{\epsilon}}\}$. Then, as the interval $(\hat{\epsilon}^2, \epsilon_0^2)$ is contractible, Lemma 4.3 tells us immediately that $G : B_\epsilon \cap G^{-1}(\partial B_\delta) \rightarrow \partial B_\delta$ and $G : B_{\epsilon'} \cap G^{-1}(\partial B_\delta) \rightarrow \partial B_\delta$ have the same diffeomorphism-type. \square

Remark 4.5. One might hope suspect that, if n and k are even, then Milnor's condition's (a) and (b) are satisfied by maps G which come from complex analytic maps. This is **not** the case.

Even in the nice case of a complex analytic isolated complete intersection singularity, the set of critical values would not locally consist solely of the origin, but would instead be a hypersurface in an open neighborhood of the origin in $\mathbb{C}^{k/2}$; see [11]. Thus, Milnor's condition (a) does not hold.

This means that the types of Milnor fibrations that we obtain when we assume Milnor's conditions (a) and (b) are extremely special.

Definition 4.6. *If G satisfies Milnor's conditions (a) and (b) at $\mathbf{0}$, we refer to the two fibrations over spheres (or their diffeomorphism classes) in Theorem 4.4, $G : B_\epsilon \cap G^{-1}(\partial B_\delta) \rightarrow \partial B_\delta$ and $G : \circ B_\epsilon \cap G^{-1}(\partial B_\delta) \rightarrow \partial B_\delta$, as the **compact Milnor fibration of G at $\mathbf{0}$ inside a ball** and the **Milnor fibration of G at $\mathbf{0}$ (inside a ball)**, respectively.*

Corollary 4.7. *Suppose that G satisfies Milnor's conditions (a) and (b) at $\mathbf{0}$, and that $k > 1$. Then, G maps an open neighborhood of the origin onto an open neighborhood of the origin.*

Proof. Recall that $G(\mathbf{0}) = \mathbf{0}$ and that G is not locally constant by assumption. As $k > 1$, $\circ B_{\delta_\epsilon} - \{\mathbf{0}\}$ is connected, and Theorem 4.4 implies that $G : \circ B_\epsilon \cap G^{-1}(\circ B_{\delta_\epsilon}) \rightarrow \circ B_{\delta_\epsilon}$ is surjective. \square

Definition 4.8. *Suppose that G satisfies Milnor's conditions (a) and (b) at $\mathbf{0}$, and that ϵ is a Milnor radius for G at $\mathbf{0}$. If $\delta > 0$, then (ϵ, δ) is a **Milnor pair for G at $\mathbf{0}$** if and only if there exists a $\hat{\delta} > \delta$ such that $G : B_\epsilon \cap G^{-1}(\circ B_{\hat{\delta}}^*) \rightarrow \circ B_{\hat{\delta}}^*$ is a stratified submersion (which is, of course, smooth and proper).*

Remark 4.9. Whenever we write that (ϵ, δ) is a Milnor pair for G at $\mathbf{0}$, we are assuming that G satisfies Milnor's conditions (a) and (b) at $\mathbf{0}$.

5 The Main Theorem

Now that we have finished our general discussion of \mathbb{L} -analytic functions and Milnor's conditions, we wish to investigate how they are related to each other.

The following proposition is trivial to conclude.

Proposition 5.1. *If G is a C^1 \mathbb{L} -map at p , then, near p , $\Sigma G \subseteq G^{-1}(G(p))$, i.e., G satisfies Milnor's condition (a) at p .*

Our goal is to prove that if G is \mathbb{L} -analytic at p , then G also satisfies Milnor's condition (b) at p , for then Theorem 4.4 will tell us that the Milnor fibrations inside a ball exist.

It will turn out that our method of proof allows us to conclude this for $k = 1, 2, 4$, and 8 , by using the normed division algebra structures in those dimensions. In fact, what we need is the following:

Definition 5.2. *Let \mathcal{W} be an open neighborhood of the origin in \mathbb{R}^k . Let P be an infinite subset of \mathbb{N} (the **admissible powers**). A function $M : P \times \mathcal{W} \rightarrow \mathbb{R}^k$, where $M(p, y)$ is written as $M_p(y)$, is a **normed analytic power map (a nap-map)** if and only if, for all $p \in P$,*

1. M_p is real analytic;
2. for all $y \in \mathcal{W}$, $|M_p(y)| = |y|^p$;

3. the image of M_p contains an open neighborhood of the origin; and
4. there exists a constant $K_p > 0$ such that, if $M_p = (m_1, \dots, m_k)$, then, for all $y \in \mathcal{W}$,

$$\max_{|(t_1, \dots, t_k)|=1} |t_1 \nabla m_1(y) + \dots + t_k \nabla m_k(y)| \leq K_p |y|^{p-1}.$$

Remark 5.3. It is easy to show that Items 1 and 2 in the above definition (or even replacing real analytic by C^1) imply that

$$\min_{|(t_1, \dots, t_k)|=1} |t_1 \nabla m_1(y) + \dots + t_k \nabla m_k(y)| \leq p |y|^{p-1} \leq \max_{|(t_1, \dots, t_k)|=1} |t_1 \nabla m_1(y) + \dots + t_k \nabla m_k(y)|.$$

We will not use these inequalities.

Originally, we used multiplication in the reals, complex numbers, quaternions, and octonions to produce nap-maps from \mathbb{R}^k to \mathbb{R}^k when $k = 1, 2, 4, \text{ or } 8$. T. Gaffney produced the easier nap-maps below **for all** k .

Proposition 5.4. *For all k , there exists a nap-map from \mathbb{R}^k to \mathbb{R}^k . In particular, if P is the set of odd natural numbers, then, for all k , the function $M : P \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ given by $M(p, y) = M_p(y) = |y|^{p-1}y$ is a nap-map.*

Proof. Let P and M_p be as in the statement of the proposition. Since $p \in P$ is odd, Items 1, 2, and 3 in the definition of a nap-map are trivially satisfied. We need to prove the inequality in Item 4 of Definition 5.2.

Let $M_p = (m_1, \dots, m_k)$. One calculates that

$$\frac{\partial m_i}{\partial y_j} = |y|^{p-1} \delta_{i,j} + (p-1) |y|^{p-3} y_i y_j,$$

where $\delta_{i,j}$ is the Kronecker delta function. Thus, considering y as a column-vector and denoting its transpose by y^t , the transpose of the derivative matrix of M_p at y is

$$L := |y|^{p-1} I + (p-1) |y|^{p-3} y^t y.$$

Now, suppose that T is a column vector of unit length in \mathbb{R}^k . Our goal is to show that $|LT| \leq K_p |y|^{p-1}$ for some constant K_p . We find

$$|LT| = \left| |y|^{p-1} T + (p-1) |y|^{p-3} y^t y T \right| \leq \left| |y|^{p-1} T \right| + \left| (p-1) |y|^{p-3} y^t y T \right| \leq |y|^{p-1} + (p-1) |y|^{p-1},$$

where we use repeatedly in the last inequality that for matrices A and B , $|AB| \leq |A| |B|$. Thus, $\max_{|(t_1, \dots, t_k)|=1} |t_1 \nabla m_1(y) + \dots + t_k \nabla m_k(y)| = |L| \leq p |y|^{p-1}$. \square

We can now prove our main lemma.

Lemma 5.5. *Suppose that G is L -analytic at $\mathbf{0}$, and $G(\mathbf{0}) = \mathbf{0}$. Then, exists an open neighborhood \mathcal{W} of $\mathbf{0}$ in \mathcal{U} and a Whitney stratification \mathfrak{S} of $\mathcal{W} \cap X$ such that, for all $S \in \mathfrak{S}$, the pair $(\mathcal{W} - X, S)$ satisfies Thom's a_G condition, i.e., if $p_j \rightarrow p \in S \in \mathfrak{S}$, where $p_j \in \mathcal{W} - X$, and $T_{p_j} G^{-1} G(p_j)$ converges to some linear subspace \mathcal{T} in the Grassmanian of $(n-k)$ -dimensional linear subspaces of \mathbb{R}^n , then $T_p S \subseteq \mathcal{T}$.*

Proof. This proof follows that of Theorem 1.2.1 of [6].

Let \mathcal{W} , c , and θ be as in Definition 3.3: \mathcal{W} is an open neighborhood of $\mathbf{0}$ in \mathcal{U} , and $c, \theta \in \mathbb{R}$ are such that $c > 0$, $0 < \theta < 1$, and, for all $x \in \mathcal{W}$,

$$|G(x)|^\theta \leq c \cdot \min_{|(a_1, \dots, a_k)|=1} |a_1 \nabla g_1(x) + \dots + a_k \nabla g_k(x)| = c \sigma_k(x).$$

Let M be a nap-map on a neighborhood \mathcal{V} of the origin in \mathbb{R}^k (which exist by Proposition 5.4), and let $P \subseteq \mathbb{N}$ denote the set of admissible powers. Let $\pi \in P$ be such that $\pi > 1/(1 - \theta)$, so that the image of M_π contains a neighborhood of the origin. Let $M_\pi = (m_1, \dots, m_k)$.

Consider the real analytic map $\widehat{G} : \mathcal{W} \times \mathcal{V} \rightarrow \mathbb{R}^k$ given by

$$\widehat{G}(x, y) := (g_1(x) + m_1(y), \dots, g_k(x) + m_k(y)).$$

Let $\widehat{\mathfrak{S}}$ be a Whitney stratification of $V(\widehat{G})$ such that $V(G) \times \{\mathbf{0}\}$ is a union of strata. Let $\mathfrak{S} := \{S \mid S \times \{\mathbf{0}\} \in \widehat{\mathfrak{S}}\}$. We claim that \mathcal{W} and \mathfrak{S} satisfy the conclusion of the lemma.

It will be convenient to deal with the conormal formulation of the a_G condition. Suppose that we have a sequence of points $p_j \rightarrow p \in S \in \mathfrak{S}$, where $p_j \in \mathcal{W} - X$, and a sequence ${}^j \mathbf{a} := ({}^j a_1, \dots, {}^j a_k) \in \mathbb{R}^k$ such that ${}^j a_1 d_{p_j} g_1 + \dots + {}^j a_k d_{p_j} g_k$ converges to a cotangent vector η (in the fiber of $T^*\mathcal{W}$ over p). We wish to show that $\eta(T_p S) \equiv 0$.

If $\eta \equiv 0$, there is nothing to show; so we assume that $\eta \neq 0$.

We may assume that, for all j , ${}^j \mathbf{a} \neq \mathbf{0}$. Also, as $p_j \rightarrow p \in V(G)$, we may assume that, for all j , $-G(p_j)$ is in the image of M_π . Let ${}^j \mathbf{u} := ({}^j u_1, \dots, {}^j u_k) \in M_\pi^{-1}(-G(p_j))$, so that $q_j := (p_j, {}^j \mathbf{u}) \in V(\widehat{G})$. Since $G(p) = 0$ and $\widehat{G}(q_j) = 0$, $M_\pi({}^j \mathbf{u}) \rightarrow \mathbf{0}$; by Item 2 in the definition of a nap-map, it follows that ${}^j \mathbf{u} \rightarrow \mathbf{0}$, and so $q_j \rightarrow (p, \mathbf{0}) \in S \times \{\mathbf{0}\} \in \widehat{\mathfrak{S}}$.

By taking a subsequence, if necessary, and using that $\eta \neq 0$, we may assume that the projective class

$$[{}^j a_1 d_{q_j} (g_1 + m_1) + \dots + {}^j a_k d_{q_j} (g_k + m_k)]$$

converges to some $[\omega]$, where $\omega = b_1 dx_1 + \dots + b_n dx_n + s_1 dy_1 + \dots + s_k dy_k \neq 0$. By Whitney's condition (a), we have that $\omega(T_p S \times \{\mathbf{0}\}) \equiv 0$.

Let us reformulate part of our discussion above using vectors, instead of covectors. In terms of vectors, we are assuming that ${}^j a_1 \nabla g_1(p_j) + \dots + {}^j a_k \nabla g_k(p_j) \rightarrow \mathbf{w} \neq \mathbf{0}$, and that

$$\left[{}^j a_1 \left(\nabla g_1(p_j), \nabla m_1({}^j \mathbf{u}) \right) + \dots + {}^j a_k \left(\nabla g_k(p_j), \nabla m_k({}^j \mathbf{u}) \right) \right] \rightarrow [(b_1, \dots, b_n, s_1, \dots, s_k)] \neq \mathbf{0}.$$

If we could show that the projective classes $[\mathbf{w} \times \{\mathbf{0}\}]$ and $[(b_1, \dots, b_n, s_1, \dots, s_k)]$ are equal, then we would be finished.

To show that $[\mathbf{w} \times \{\mathbf{0}\}]$ and $[(b_1, \dots, b_n, s_1, \dots, s_k)]$ are equal, it clearly suffices to show that

$$\frac{|{}^j a_1 \nabla m_1({}^j \mathbf{u}) + \dots + {}^j a_k \nabla m_k({}^j \mathbf{u})|}{|{}^j a_1 \nabla g_1(p_j) + \dots + {}^j a_k \nabla g_k(p_j)|} \rightarrow 0.$$

Dividing the numerator and denominator by $|{}^j \mathbf{a}|$, we see that we may assume that $|{}^j \mathbf{a}| = 1$.

Using that G is an L-map at $\mathbf{0}$ and Item 4 in the definition of a nap-map, we obtain that

$$\frac{|j a_1 \nabla m_1(j \mathbf{u}) + \cdots + j a_k \nabla m_k(j \mathbf{u})|}{|j a_1 \nabla g_1(p_j) + \cdots + j a_k \nabla g_k(p_j)|} \leq \frac{K_p |j \mathbf{u}|^{\pi-1}}{|G(p_j)|^\theta}.$$

Now, $|j \mathbf{u}|^\pi = |M_\pi(j \mathbf{u})| = |G(p_j)|$. Thus, $|j \mathbf{u}|^{\pi-1} = |G(p_j)|^{(\pi-1)/\pi}$, and we would like to show that

$$\frac{|G(p_j)|^{(\pi-1)/\pi}}{|G(p_j)|^\theta} \rightarrow 0.$$

However, this follows at once, since $G(p_j) \rightarrow \mathbf{0}$ and, by our choice of π , $(\pi - 1)/\pi > \theta$. \square

Remark 5.6. In Lemma 5.5, we assumed that G was real analytic and used that a normed, analytic power map M_p exists on \mathbb{R}^k . We assumed analyticity for both maps so that $V(\widehat{G})$ would have a Whitney stratification. However, it is enough to assume that G and M_p are *subanalytic*. See [7].

The main theorem follows easily. Note that, when $k = 1$, Theorem 1.1 implies that, if G is real analytic, then G is L-analytic.

Theorem 5.7. *Suppose that G is L-analytic at p . Then, Milnor's condition's (a) and (b) hold at p .*

Proof. We assume, without loss of generality, that $p = \mathbf{0}$ and $G(\mathbf{0}) = \mathbf{0}$.

Milnor's condition (a) is immediate, as we stated in Proposition 5.1. Milnor's condition (b) follows from Lemma 5.5 by using precisely the argument of Lê in [9].

Let $\epsilon_0 > 0$ be a Milnor (a) radius for G at $\mathbf{0}$ such that, for all ϵ' such that $0 < \epsilon' \leq \epsilon_0$, the sphere $S_{\epsilon'}$ transversely intersects all of the strata of the Whitney a_G stratification whose existence is guaranteed by Lemma 5.5. We claim that ϵ_0 is a Milnor (b) radius.

Suppose not. Then, there would be a sequence of points $p_i \in B_{\epsilon_0} - V(G)$ such that $G(p_i) \rightarrow 0$ and $T_{p_i}(G^{-1}G(p_i)) \subseteq T_{p_i}S_{\epsilon_i}$, where ϵ_i denotes the distance of p_i from the origin (we use Milnor's condition (a) here to know that $G^{-1}G(p_i)$ is smooth). By taking a subsequence, we may assume that the p_i approach a point $p \in S_\epsilon \cap X$ and that $T_{p_i}(G^{-1}G(p_i))$ approaches a limit \mathcal{T} . If we let M denote the stratum of our a_G stratification which contains p , then we find that $T_p M \subseteq \mathcal{T}$ and $\mathcal{T} \subseteq T_p S_\epsilon$. As S_ϵ transversely intersects M , we have a contradiction, which proves the desired result. \square

Corollary 5.8. *Suppose that G is L-analytic at p . Then, the Milnor fibrations for $G - G(p)$ inside a ball, centered at p , exist.*

Proof. As we saw in Section 4, this follows immediately from Milnor's condition's (a) and (b). \square

We include the following just to emphasize that L-analytic have many properties that one associates with complex analytic functions.

Corollary 5.9. *Suppose that G is an L-analytic function which is nowhere locally constant, and that $k \geq 2$. Then, G is an open map.*

Proof. This follows at once from Theorem 5.7 and Corollary 4.7. \square

6 Comments and Questions

We have many basic questions.

Question 6.1. Are there real analytic maps which are \mathbb{L} -maps but do not have positive \mathbb{L} -weight?

We suspect that the answer to the above question is “yes”.

Given Theorem 3.13, it is natural to ask:

Question 6.2. Is the composition of two \mathbb{L} -maps an \mathbb{L} -map? What if the maps are \mathbb{L} -analytic?

Considering Example 3.15, we are led to ask:

Question 6.3. If one takes two simple \mathbb{L} -maps, in disjoint variables, into \mathbb{R}^4 or \mathbb{R}^8 and multiplies them using quaternionic or octonionic multiplication (analogous to the complex situation in Example 3.15), does one obtain a new simple \mathbb{L} -map?

We feel certain that the answer to the above question is “yes”, but we have not actually verified this.

It is not difficult to produce examples of real analytic $f = (g, h)$ which possess Milnor fibrations inside balls, and yet are **not** \mathbb{L} -analytic. It is our hope that the \mathbb{L} -analytic condition is strong enough to allow one to conclude that the Milnor fibration inside an open ball is equivalent to the Milnor fibration **on the sphere** (the fibration analogous to $f_c/|f_c| : S_\epsilon - S_\epsilon \cap X \rightarrow S^1 \subseteq \mathbb{C}$, which exists when f_c is complex analytic).

As we discussed in the introduction, in the real analytic setting, $f/|f|$ does not necessarily yield the projection map of a fibration from $S_\epsilon - S_\epsilon \cap X$ to S^1 ; when this map is, in fact, a locally trivial fibration (for all sufficiently small ϵ), one says that f satisfies the **strong Milnor condition** at the origin (see VII.2 of [19]).

It is not terribly difficult to try to mimic Milnor’s proof that the Milnor fibration inside the ball and the one on the sphere are diffeomorphic; one wants to integrate an appropriate vector field. In the case where $f = (g, h)$ is real analytic and satisfies Milnor’s conditions (a) and (b), one finds that one needs a further condition:

Definition 6.4. Let $\omega := -h\nabla g + g\nabla h$, and assume that $f(\mathbf{0}) = \mathbf{0}$. We say that f **satisfies Milnor’s condition (c) at $\mathbf{0}$** if and only if there exists an open neighborhood \mathcal{W} of $\mathbf{0}$ in \mathcal{U} such that, at all points \mathbf{x} of $\mathcal{W} - X$, if \mathbf{x} is a linear combination of ∇g and ∇h , then

$$|\omega(\mathbf{x})|^2(\mathbf{x} \cdot \nabla|f|^2(\mathbf{x})) > (\nabla|f|^2(\mathbf{x}) \cdot \omega(\mathbf{x}))(\mathbf{x} \cdot \omega(\mathbf{x})).$$

It is not too difficult to show that, if f is a simple \mathbb{L} -analytic function, there exists an open neighborhood \mathcal{W} of $\mathbf{0}$ in \mathcal{U} such that, at all points \mathbf{x} of $\mathcal{W} - X$, $\mathbf{x} \cdot \nabla|f|^2(\mathbf{x}) > 0$. It is also trivial that a simple \mathbb{L} -analytic function has $\nabla|f|^2(\mathbf{x}) \cdot \omega(\mathbf{x}) = 0$. Milnor’s condition (a) implies that \mathcal{W} can be chosen so that $\omega(\mathbf{x}) \neq 0$ for $\mathbf{x} \in \mathcal{W} - X$. Thus, we conclude that simple \mathbb{L} -analytic functions satisfy the strong Milnor condition.

However, the real question is:

Question 6.5. Do general \mathbb{L} -analytic maps into \mathbb{R}^2 satisfy Milnor's condition (c), and so satisfy the strong Milnor condition? What about real analytic maps with positive \mathbb{L} -weight? In light of Proposition 3.20, what about real analytic maps with positive \mathbb{L} -weight where ∇g and ∇h have comparable in magnitude?

Finally, having seen in this paper that \mathbb{L} -analytic pairs of functions share some important properties with complex analytic functions, one is led to ask a very general question:

Question 6.6. Do \mathbb{L} -analytic maps form an interesting class of functions to study for reasons having nothing to do with Milnor fibrations?

References

- [1] Bierstone, E. and Milman, P. Semianalytic and subanalytic sets. *Publ. math. de I.H.É.S.*, 67:5–42, 1988.
- [2] Dimca, A. *Singularities and Topology of Hypersurfaces*. Universitext. Springer-Verlag, 1992.
- [3] dos Santos, R. Topological triviality of families of real isolated singularities and their Milnor fibrations. *Math. Scand.*, 96, no. 1:96–106, 2005.
- [4] Ehresmann, C. Les connexions infinitésimales dans un espace fibré différentiable. *Colloque de Topologie, Bruxelles*, pages 29–55, 1950.
- [5] Gaffney, T. Integral closure of modules and Whitney equisingularity. *Invent. Math.*, 107:301–322, 1992.
- [6] Hamm, H. and Lê D. T. Un théorème de Zariski du type de Lefschetz. *Ann. Sci. Éc. Norm. Sup.*, 6 (series 4):317–366, 1973.
- [7] Hardt, R. Topological properties of subanalytic sets. *Trans. AMS*, 211:57–70, 1975.
- [8] Jacquemard, A. Fibrations de Milnor pour des applications réelles. *Boll. Un. Mat. Ital.B. Serie VII*, 3:591–600, 1989.
- [9] Lê, D. T. Some remarks on Relative Monodromy. In P. Holm, editor, *Real and Complex Singularities, Oslo 1976*, pages 397–404. Nordic Summer School/NAVF, 1977.
- [10] Lojasiewicz, S. *Ensemble semi-analytiques*. IHES Lecture notes, 1965.
- [11] Looijenga, E. *Isolated Singular Points on Complete Intersections*. Cambridge Univ. Press, 1984.
- [12] Massey, D. *Lê Cycles and Hypersurface Singularities*, volume 1615 of *Lecture Notes in Math*. Springer-Verlag, 1995.
- [13] Mather, J. Notes on Topological Stability. Notes from Harvard Univ., 1970.
- [14] Milnor, J. *Singular Points of Complex Hypersurfaces*, volume 77 of *Annals of Math. Studies*. Princeton Univ. Press, 1968.
- [15] Pichon, A. and Seade, J. Real singularities and open-book decompositions of the 3-sphere. *Annales de la faculté des sciences de Toulouse, Sér. 6*, 12, no. 2:245–265, 2003.
- [16] Randell, R. On the Topology of Non-isolated Singularities. *Proc. Georgia Top. Conf., Athens, Ga., 1977*, pages 445–473, 1979.
- [17] Ruas, M. A. S., Seade, J., and Verjovsky, A. *On real singularities with a Milnor fibration*, pages 191–213. Birkhäuser, 2002.
- [18] Ruas, M. and dos Santos, R. Real Milnor fibrations and (c)-regularity. *Manuscripta Math.*, 17, no. 2:207–218, 2005.
- [19] Seade, J. *On the topology of isolated singularities in analytic spaces*, volume 241 of *Progress in Mathematics*. Birkhäuser, 2005.